SOME RESULTS ON COMMON FIXED POINTS FOR RATIONAL TYPE
CONTRACTION MAPPINGS IN COMPLEX VALUED METRIC SPACE

DEEPAK KUMAR* AND AMAL CHACKO

Abstract. In this manuscript, we have obtained the sufficient conditions for the existence
and uniqueness of a pair of mappings satisfying rational type contractive conditions in the
framework of complex valued metric space. Our result generalizes the well known result
introduced by Azam et al. [2] in complex valued metric space. Also, various deductions
have been provided.

1. Introduction

Azam et al. [2] introduced the concept of more general metric space, which is well known as
complex valued metric spaces. He gave sufficient conditions for the existence and uniqueness
of common fixed points satisfying contractive conditions. Later, S. Bhatt et al. [1] without
using the notion of continuity proved a common fixed point theorem for weakly compatible
type contractive conditions proved some common fixed point theorems in the framework
fixed-point theorems for two single-valued mappings satisfy certain metric inequalities.

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* Corresponding author
The notion of complex valued metric space lead to development in non linear analysis. Thereafter, many results have been proved by the researchers in the framework of complex valued metric spaces for references (see [7]-[13]).

2. Preliminaries

To begin with, we recall some basic definitions, notations, and results. The following definitions of Azam et al. [2] are required in the sequel.

Let $C$ be a set of complex number such that $z_1, z_2 \in C$. Define a partial order $\preccurlyeq$ on $C$, such that $z_1 \preccurlyeq z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2), \text{Img}(z_1) \leq \text{Img}(z_2)$.

It follows that $z_1 \preccurlyeq z_2$ if one of the below mentioned conditions is satisfied:

1. $(i)$ $\text{Re}(z_1) = \text{Re}(z_2), \text{Img}(z_1) < \text{Img}(z_2)$;
2. $(ii)$ $\text{Re}(z_1) < \text{Re}(z_2), \text{Img}(z_1) = \text{Img}(z_2)$;
3. $(iii)$ $\text{Re}(z_1) < \text{Re}(z_2), \text{Img}(z_1) < \text{Img}(z_2)$;
4. $(iv)$ $\text{Re}(z_1) = \text{Re}(z_2), \text{Img}(z_1) = \text{Img}(z_2)$.

In particular, we will write $z_1 \preccurlyeq z_2$, if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied. We will write $z_1 \prec z_2$ if only (iii) is satisfied.

**Remark 2.1.** We obtained that the following statements holds:

- $a, b \in \mathbb{R}$ and $a \leq b$ implies $az \preccurlyeq bz$, for all $z \in \mathbb{C}$;
- $0 \preccurlyeq z_1 \preccurlyeq z_2$ implies $|z_1| < |z_2|$;
- $z_1 \preccurlyeq z_2$ and $z_2 \prec z_3$ imply $z_1 \prec z_3$.

**Definition 2.1.** [2] Let $X$ be non-empty set. Suppose that the mapping $\rho_c : X \times X \to \mathbb{C}$ satisfies the following conditions:

1. $(i)$ $0 \preccurlyeq \rho_c(x, y)$ for all $x, y \in X$ and $\rho_c(x, y) = 0$ if $x = y$;
2. $(ii)$ $\rho_c(x, y) = \rho_c(y, x)$ for all $x, y \in X$;
3. $(iii)$ $\rho_c(x, y) \preccurlyeq \rho_c(x, z) + \rho_c(z, y)$ for all $x, y, z \in X$.

Then, $\rho_c$ is called a complex valued metric on $X$, and $(X, \rho_c)$ is called complex valued metric space.

**Definition 2.2.** [2] A point $x \in X$ is called an interior of a set $A \subseteq X$ whenever their exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : \rho_c(x, y) \prec r\} \subseteq A$. 
Definition 2.3. Let \( \{x_n\} \) be a sequence in \( X \) and \( x \in X \). If for every \( c \in \mathbb{C} \) with \( 0 < c \), there is \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \), \( \rho_c(x_n, x) < c \), then \( \{x_n\} \) is said to be convergent, \( \{x_n\} \) converges to \( x \) and \( x \) is the limit of \( \{x_n\} \). We denote this by \( \lim_{n \to \infty} x_n = x \). If for every \( c \in \mathbb{C} \) with \( 0 < c \) there is \( n_0 \in \mathbb{N} \), such that for all \( n > n_0 \), \( \rho_c(x_m, x_{n+m}) < c \), then \( \{x_n\} \) is called a Cauchy sequence in \( (X, \rho_c) \).

Definition 2.4. If every Cauchy sequence is convergent in \( (X, \rho_c) \) then \( (X, \rho_c) \) is called a complete complex valued metric space.

Lemma 2.1. Let \( (X, \rho_c) \) be a complex valued metric space and let \( \{x_n\} \) be a sequence in \( X \). Then, \( \{x_n\} \) converges to \( x \) if and only if \( |\rho_c(x_n, x)| \to 0 \) as \( n \to \infty \).

Lemma 2.2. Let \( (X, \rho_c) \) be a complex valued metric space and let \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) is a Cauchy sequence if and only if \( |\rho_c(x_n, x_{n+m})| \to 0 \) as \( n \to \infty \).

3. Some results on Fixed point

Theorem 3.1. Let \( (X, \rho_c) \) be a complete complex valued metric space and \( S, T : X \to X \) be self mappings satisfying the following condition:

\[
\rho_c(Sx, Ty) \preceq \alpha \rho_c(x, y) + \beta \frac{\rho_c(x, Sx)\rho_c(y, Ty)}{1 + \rho_c(x, y)} + \gamma \frac{\rho_c(x, Sx)\rho_c(y, Ty)}{1 + \rho_c(x, y) + \rho_c(x, Ty) + \rho_c(y, Sx)}
\]

for all \( x, y \in X \), where \( \alpha, \beta, \gamma \) are non-negative reals with \( \alpha + \beta + \gamma < 1 \). Then \( S, T \) have a unique common fixed point.

Proof. Let \( x_0 \in X \) be any arbitrary point and define \( x_{2k+1} = Sx_{2k} \) and \( x_{2k+2} = Tx_{2k+1} \).

Then,

\[
\rho_c(x_{2k+1}, x_{2k+2}) = \rho_c(Sx_{2k}, Tx_{2k+1})
\]

\[
\preceq \alpha \rho_c(x_{2k}, x_{2k+1}) + \beta \frac{\rho_c(x_{2k}, Sx_{2k})\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k}, x_{2k+1})}
\]

\[
+ \gamma \frac{\rho_c(x_{2k}, Sx_{2k})\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k}, Tx_{2k+1}) + \rho_c(x_{2k+1}, Sx_{2k})}
\]

\[
\preceq \alpha \rho_c(x_{2k}, x_{2k+1}) + \beta \frac{\rho_c(x_{2k}, x_{2k+1})\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k}, x_{2k+1})}
\]

\[
+ \gamma \frac{\rho_c(x_{2k}, x_{2k+1})\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k}, x_{2k+2}) + \rho_c(x_{2k+1}, x_{2k+2})}
\]

...
Since,

\[ \rho_c(x_{2k}, x_{2k+1}) \leq 1 + \rho_c(x_{2k}, x_{2k+1}) \] and

\[ \rho_c(x_{2k}, x_{2k+1}) \leq 1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k}, x_{2k+2}). \]

Therefore,

\[ \rho_c(x_{2k+1}, x_{2k+2}) \lesssim \alpha \rho_c(x_{2k}, x_{2k+1}) + \beta \rho_c(x_{2k+1}, x_{2k+2}) + \gamma \rho_c(x_{2k+1}, x_{2k+2}) \]

\[ \rho_c(x_{2k+1}, x_{2k+2}) \lesssim \frac{\alpha}{1 - \beta - \gamma} \rho_c(x_{2k}, x_{2k+1}). \]

Similarly,

\[ \rho_c(x_{2k+2}, x_{2k+3}) = \rho_c(x_{2k+3}, x_{2k+2}) = \rho_c(Sx_{2k+2}, Tx_{2k+1}) \]

\[ \lesssim \alpha \rho_c(x_{2k+2}, x_{2k+1}) + \beta \rho_c(x_{2k+2}, Sx_{2k+2}) \rho_c(x_{2k+1}, Tx_{2k+1}) \]

\[ + \gamma \frac{\rho_c(x_{2k+2}, Sx_{2k+2}) \rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+2}, x_{2k+1}) + \rho_c(x_{2k+2}, Sx_{2k+2}) + \rho_c(x_{2k+1}, Sx_{2k+2})} \]

\[ \lesssim \alpha \rho_c(x_{2k+2}, x_{2k+1}) + \beta \rho_c(x_{2k+2}, x_{2k+3}) \rho_c(x_{2k+1}, x_{2k+2}) \]

\[ + \gamma \frac{\rho_c(x_{2k+2}, x_{2k+3}) \rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+2}, x_{2k+1}) + \rho_c(x_{2k+2}, x_{2k+2}) + \rho_c(x_{2k+1}, x_{2k+3})} \]

Since,

\[ \rho_c(x_{2k+2}, x_{2k+1}) \leq 1 + \rho_c(x_{2k+2}, x_{2k+1}) \] and

\[ \rho_c(x_{2k+2}, x_{2k+1}) \leq 1 + \rho_c(x_{2k+2}, x_{2k+1}) + \rho_c(x_{2k+1}, x_{2k+3}). \]

Therefore,

\[ \rho_c(x_{2k+2}, x_{2k+3}) \lesssim \alpha \rho_c(x_{2k+2}, x_{2k+1}) + \beta \rho_c(x_{2k+2}, x_{2k+3}) + \gamma \rho_c(x_{2k+2}, x_{2k+3}) \]

\[ \rho_c(x_{2k+2}, x_{2k+3}) \lesssim \frac{\alpha}{1 - \beta - \gamma} \rho_c(x_{2k+2}, x_{2k+1}) \]

or

\[ \rho_c(x_{2k+2}, x_{2k+3}) \lesssim \frac{\alpha}{1 - \beta - \gamma} \rho_c(x_{2k+1}, x_{2k+2}). \]

Assume, \( h = \frac{\alpha}{1 - \beta - \gamma} < 1 \), we have

\[ \rho_c(x_{n+1}, x_{n+2}) \lesssim hd(x_n, x_{n+1}) \lesssim \ldots \lesssim h^{n+1} \rho_c(x_0, x_1). \]
For some $m > n$, we have

\[
\rho_c(x_n, x_m) \leq \rho_c(x_n, x_{n+1}) + \rho_c(x_{n+1}, x_{n+2}) + \ldots + \rho_c(x_{m-1}, x_m)
\]

\[
\leq [h^n + h^{n+1} + \ldots + h^{m-1}]\rho_c(x_0, x_1)
\]

\[
= \left[ \frac{h^n}{1 - h} \right] \rho_c(x_0, x_1).
\]

This implies,

\[
|\rho_c(x_m, x_n)| \leq \left[ \frac{h^n}{1 - h} \right] |\rho_c(x_0, x_1)| \to 0, \text{ as } m, n \to \infty.
\]

Hence, \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete complex valued metric space, therefore there exists \( u \in X \) such that \( x_n \to u \), we shall show that \( u = Su \). To prove that \( \rho_c(u, Su) = z > 0 \). Therefore, by using triangle inequality, we have

\[
\rho_c(u, Su) = z \leq \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+2}, Su)
\]

\[
\leq \rho_c(u, x_{2k+2}) + \rho_c(Tx_{2k+1}, Su)
\]

\[
\leq \rho_c(u, x_{2k+2}) + \alpha \rho_c(x_{2k+1}, u) + \beta \frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u)}
\]

\[
+ \gamma \frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u) + \rho_c(x_{2k+1}, Su) + \rho_c(u, Tx_{2k+1})}
\]

\[
\leq \rho_c(u, x_{2k+2}) + \alpha \rho_c(x_{2k+1}, u) + \beta \frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u)}
\]

\[
+ \gamma \frac{zd(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u) + \rho_c(x_{2k+1}, u) + \rho_c(u, x_{2k+2})}.
\]

This implies,

\[
|\rho_c(u, Su)| \leq |\rho_c(u, x_{2k+2})| + \alpha |\rho_c(x_{2k+1}, u)| + \beta \frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{1 + \rho_c(x_{2k+1}, u)}
\]

\[
+ \gamma \frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{1 + \rho_c(x_{2k+1}, u) + \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+1}, u)}.
\]

Letting \( k \to \infty \), we have \( |\rho_c(u, Su)| \leq 0 \), hence \( \rho_c(u, Su) = 0 \). That is \( z = 0 \), a contradiction. Hence our supposition is wrong. Therefore, \( z = 0 \), ie \( Su = u \). On the same lines, we can show that \( u = Tu \). Therefore, \( u \) is a common fixed point of \( S \) and \( T \).

Now, we shall show that \( u \) is a unique common fixed point of \( S \) and \( T \). For this, Consider \( u^* = u \) be another common fixed point of \( S \) and \( T \).
Therefore,
\[
\rho_c(u, u^*) = \rho_c(Su, Tu^*)
\]
\[
\lesssim \alpha \rho_c(u, u^*) + \beta \frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, u^*)} + \gamma \frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, u^*)} + \rho_c(u, Tu^*) + \rho_c(u^*, Su)
\]
\[
\lesssim \alpha \rho_c(u, u^*).
\]

This implies \((1 - \alpha)\rho_c(u, u^*) \lesssim 0\) and hence, \((1 - \alpha)|\rho_c(u, u^*)| \leq 0\).

Therefore, \(\rho_c(u, u^*) = 0\) and hence, \(u = u^*\), which implies uniqueness. Thus \(u\) is a unique common fixed point of \(S\) and \(T\).

**Corollary 3.1.** Let \((X, \rho_c)\) be a complete complex valued metric space and \(T : X \to X\) be a self mapping satisfying the following condition:
\[
\rho_c(Tx, Ty) \lesssim \alpha \rho_c(x, y) + \beta \frac{\rho_c(x, Tx)\rho_c(y, Ty)}{1 + \rho_c(x, y)} + \gamma \frac{\rho_c(x, Tx)\rho_c(y, Ty)}{1 + \rho_c(x, y)} + \rho_c(x, Tx) + \rho_c(y, Ty)
\]
for all \(x, y \in X\), where \(\alpha, \beta, \gamma\) are non-negative reals with \(\alpha + \beta + \gamma < 1\). Then \(T\) has a unique fixed point.

**Corollary 3.2.** Let \((X, \rho_c)\) be a complete complex valued metric space and \(T : X \to X\) be a self mapping satisfying the following condition:
\[
\rho_c(T^n x, T^n y) \lesssim \alpha \rho_c(x, y) + \beta \frac{\rho_c(x, T^n x)\rho_c(y, T^n y)}{1 + \rho_c(x, y)} + \gamma \frac{\rho_c(x, T^n x)\rho_c(y, T^n y)}{1 + \rho_c(x, y)} + \rho_c(x, T^n x) + \rho_c(y, T^n x)
\]
for all \(x, y \in X\), where \(\alpha, \beta, \gamma\) are non-negative reals with \(\alpha + \beta + \gamma < 1\). Then \(T\) has a unique fixed point.

**Proof.** By Corollary 3.1, we obtain \(\eta \in X\) such that \(T^n \eta = \eta\).

The result then follows from the fact that,
\[
\rho_c(T^n \eta, \eta) = \rho_c(T^n \eta, T^n \eta) = \rho_c(T^n \eta, T^n \eta)
\]
\[
\lesssim \alpha \rho_c(T^n \eta, \eta) + \beta \frac{\rho_c(T^n \eta, T^n \eta)}{1 + \rho_c(T^n \eta, \eta)} + \gamma \frac{\rho_c(T^n \eta, T^n \eta)}{1 + \rho_c(T^n \eta, \eta)} + \rho_c(T^n \eta, T^n \eta) + d(\eta, T^n \eta)
\]
\[
\lesssim \alpha \rho_c(T^n \eta, \eta) + \beta \frac{\rho_c(T^n \eta, T^n \eta)}{1 + \rho_c(T^n \eta, \eta)} + \gamma \frac{\rho_c(T^n \eta, T^n \eta)}{1 + \rho_c(T^n \eta, \eta)} + \rho_c(T^n \eta, T^n \eta) + d(\eta, T^n \eta)
\]
\[
= \alpha \rho_c(T^n \eta, \eta)
\]
Therefore, \((1 - \alpha)\rho_c(T\eta, \eta) \lesssim 0\), this implies, \((1 - \alpha)|\rho_c(T\eta, \eta)| \leq 0\), hence \(\rho_c(T^n\eta, \eta) = 0\). Thus, \(\eta\) is a fixed point of \(T\). On the same lines of Theorem 3.1, we can prove the uniqueness.

**Theorem 3.2.** Let \((X, \rho_c)\) be a complete complex valued metric space and \(S, T : X \to X\) be self mappings satisfying the following condition:

\[
\rho_c(Sx, Ty) \lesssim \alpha\rho_c(x, y) + \beta \frac{\rho_c(x, Sx)\rho_c(y, Ty)}{1 + \rho_c(x, Sx)} + \gamma \frac{\rho_c(x, Sx)\rho_c(y, Ty)}{1 + \rho_c(x, Sx) + \rho_c(y, Ty)}
\]

for all \(x, y \in X\), where \(\alpha, \beta, \gamma\) are non-negative reals with \(\alpha + \beta + \gamma < 1\). Then \(S, T\) have a unique common fixed point.

**Proof.** Let \(x_0 \in X\) be any arbitrary point and define \(x_{2k+1} = Sx_{2k}\) and \(x_{2k+2} = Tx_{2k+1}\). Then,

\[
\rho_c(x_{2k+1}, x_{2k+2}) = \rho_c(Sx_{2k}, Tx_{2k+1}) \\
\lesssim \alpha \rho_c(x_{2k}, x_{2k+1}) + \beta \frac{\rho_c(x_{2k}, Sx_{2k})\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k}, Sx)} \\
+ \gamma \frac{\rho_c(x_{2k}, Sx_{2k})\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k}, Sx) + \rho_c(x_{2k+1}, Tx_{2k+1})} \\
\lesssim \alpha \rho_c(x_{2k}, x_{2k+1}) + \beta \frac{\rho_c(x_{2k}, x_{2k+1})\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k}, x_{2k+1})} \\
+ \gamma \frac{\rho_c(x_{2k}, x_{2k+1})\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k+1}, x_{2k+2})}.
\]

Following cases arises,

**Case 1.** If,

\[
\rho_c(x_{2k}, x_{2k+1}) \leq 1 + \rho_c(x_{2k}, x_{2k+1}) \quad \text{and} \quad \rho_c(x_{2k+1}, x_{2k+2}) \leq 1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k+1}, x_{2k+2}).
\]

Therefore,

\[
\rho_c(x_{2k+1}, x_{2k+2}) \lesssim \alpha \rho_c(x_{2k}, x_{2k+1}) + \beta \rho_c(x_{2k+1}, x_{2k+2}) + \gamma \rho_c(x_{2k}, x_{2k+1}) \\
\rho_c(x_{2k+1}, x_{2k+2}) \lesssim \frac{\alpha + \gamma}{1 - \beta} \rho_c(x_{2k}, x_{2k+1})
\]
Similarly,

\[ \rho_c(x_{2k+2}, x_{2k+3}) = \rho_c(Sx_{2k+2}, Tx_{2k+1}) \]

\[ \leq \alpha \rho_c(x_{2k+2}, x_{2k+1}) + \beta \frac{\rho_c(x_{2k+2}, Sx_{2k+2}) \rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+2}, x_{2k+1})} \]

\[ + \gamma \frac{\rho_c(x_{2k+2}, Sx_{2k+2}) \rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+2}, Sx_{2k+2}) + \rho_c(x_{2k+1}, Tx_{2k+1})} \]

\[ \leq \alpha \rho_c(x_{2k+2}, x_{2k+1}) + \beta \frac{\rho_c(x_{2k+2}, x_{2k+3}) \rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+2}, x_{2k+1})} \]

\[ + \gamma \frac{\rho_c(x_{2k+2}, x_{2k+3}) \rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+2}, x_{2k+3}) + \rho_c(x_{2k+1}, x_{2k+2})} \]

Since,

\[ \rho_c(x_{2k+2}, x_{2k+1}) \leq 1 + \rho_c(x_{2k+2}, x_{2k+1}) \text{ and } \]

\[ \rho_c(x_{2k+2}, x_{2k+3}) \leq 1 + \rho_c(x_{2k+2}, x_{2k+3}) + \rho_c(x_{2k+1}, x_{2k+2}). \]

Therefore,

\[ \rho_c(x_{2k+2}, x_{2k+3}) \leq \alpha \rho_c(x_{2k+1}, x_{2k+2}) + \beta \rho_c(x_{2k+2}, x_{2k+3}) + \gamma \rho_c(x_{2k+1}, x_{2k+2}) \]

\[ \rho_c(x_{2k+2}, x_{2k+3}) \leq \frac{\alpha + \gamma}{1 - \beta} \rho_c(x_{2k+1}, x_{2k+2}). \]

Assume, \( h = \frac{\alpha + \gamma}{1 - \beta} < 1 \), we have

\[ \rho_c(x_{n+1}, x_{n+2}) \leq hd(x_n, x_{n+1}) \leq \ldots \leq h^{n+1} \rho_c(x_0, x_1). \]

For some \( m > n \), we have

\[ \rho_c(x_{n}, x_{m}) \leq \rho_c(x_n, x_{n+1}) + \rho_c(x_{n+1}, x_{n+2}) + \ldots + \rho_c(x_{m-1}, x_{m}) \]

\[ \leq [h^n + h^{n+1} + \ldots + h^{m-1}] \rho_c(x_0, x_1) \]

\[ \leq \left[ \frac{h^n}{1 - h} \right] \rho_c(x_0, x_1). \]

This implies,

\[ |\rho_c(x_m, x_n)| \leq \left[ \frac{h^n}{1 - h} \right] |\rho_c(x_0, x_1)| \to 0, \text{ as } m, n \to \infty. \]

Hence, \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete complex valued metric space, therefore there exists \( u \in X \) such that \( x_n \to u \), we shall show that \( u = Su \). To prove, consider
\( \rho_c(u, Su) = z > 0 \). Therefore, by using triangle inequality, we have

\[
\rho_c(u, Su) = z \geq \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+2}, Su)
\]

\[
\geq \rho_c(u, x_{2k+2}) + \rho_c(T x_{2k+1}, Su)
\]

\[
\geq \rho_c(u, x_{2k+2}) + \alpha \rho_c(x_{2k+1}, u) + \beta \frac{\rho_c(u, Su) \rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u)} + \gamma \frac{\rho_c(u, Su) \rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u)}
\]

\[
\geq \rho_c(u, x_{2k+2}) + \alpha \rho_c(x_{2k+1}, u) + \beta \frac{z \rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u)} + \gamma \frac{z \rho_c(x_{2k+1}, x_{2k+2})}{1 + z + \rho_c(x_{2k+1}, x_{2k+2})}.
\]

This implies,

\[
|\rho_c(u, Su)| \geq |\rho_c(u, x_{2k+2})| + \alpha |\rho_c(x_{2k+1}, u)| + \beta \frac{|z| |\rho_c(x_{2k+1}, x_{2k+2})|}{1 + \rho_c(x_{2k+1}, u)} + \gamma \frac{|z| |\rho_c(x_{2k+1}, x_{2k+2})|}{1 + z + \rho_c(x_{2k+1}, x_{2k+2})}.
\]

Letting \( k \to \infty \), we have \( |\rho_c(u, Su)| \leq 0 \), hence \( \rho_c(u, Su) = 0 \). That is \( z = 0 \), a contradiction.

Hence our supposition is wrong. Therefore, \( z = 0 \), ie \( Su = u \). On the same lines, we can show that \( u = Tu \). Therefore \( u \) is a common fixed point of \( S \) and \( T \).

Now, we shall show that \( u \) is a unique common fixed point of \( S \) and \( T \). For this, Consider \( u^* = u \) be another common fixed point of \( S \) and \( T \).

Therefore,

\[
\rho_c(u, u^*) = \rho_c(Su, Tu^*)
\]

\[
\geq \alpha \rho_c(u, u^*) + \beta \frac{\rho_c(u, Su) \rho_c(u, Tu^*)}{1 + \rho_c(u, u^*)} + \gamma \frac{\rho_c(u, Su) \rho_c(u, Tu^*)}{1 + \rho_c(u, Su) + \rho_c(u, Tu^*)}
\]

\[
\geq \alpha \rho_c(u, u^*).
\]

This implies \( (1 - \alpha) \rho_c(u, u^*) \geq 0 \) and hence, \( (1 - \alpha)|\rho_c(u, u^*)| \leq 0 \).

Therefore, \( \rho_c(u, u^*) = 0 \) and hence, \( u = u^* \), which implies uniqueness. Thus, \( u \) is a unique common fixed point of \( S \) and \( T \).

**Case 2.** If,

\[
\rho_c(x_{2k}, x_{2k+1}) \leq 1 + \rho_c(x_{2k}, x_{2k+1}) \text{ and }
\]

\[
\rho_c(x_{2k}, x_{2k+1}) \leq 1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k+1}, x_{2k+2}).
\]
Therefore,

\[ \rho_c(x_{2k+1}, x_{2k+2}) \lesssim \alpha \rho_c(x_{2k}, x_{2k+1}) + \beta \rho_c(x_{2k+1}, x_{2k+2}) + \gamma \rho_c(x_{2k+1}, x_{2k+2}) \]

\[ \rho_c(x_{2k+1}, x_{2k+2}) \lesssim \frac{\alpha}{1 - \beta - \gamma} \rho_c(x_{2k}, x_{2k+1}). \]

Similarly,

\[ \rho_c(x_{2k+2}, x_{2k+3}) = \rho_c(Sx_{2k+2}, Tx_{2k+1}) \]

\[ \lesssim \alpha \rho_c(x_{2k+2}, x_{2k+1}) + \beta \frac{\rho_c(x_{2k+2}, Sx_{2k+2}) \rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+2}, x_{2k+1})} + \gamma \frac{\rho_c(x_{2k+2}, Sx_{2k+2}) \rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+2}, x_{2k+1})} \]

\[ \lesssim \alpha \rho_c(x_{2k+2}, x_{2k+1}) + \beta \frac{\rho_c(x_{2k+2}, x_{2k+3}) \rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+2}, x_{2k+1})} + \gamma \frac{\rho_c(x_{2k+2}, x_{2k+3}) \rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+2}, x_{2k+1})} \]

Since,

\[ \rho_c(x_{2k+2}, x_{2k+1}) \leq 1 + \rho_c(x_{2k+2}, x_{2k+1}) \text{ and } \]

\[ \rho_c(x_{2k+2}, x_{2k+1}) \leq 1 + \rho_c(x_{2k+2}, x_{2k+3}) + \rho_c(x_{2k+1}, x_{2k+2}). \]

Therefore,

\[ \rho_c(x_{2k+2}, x_{2k+3}) \lesssim \alpha \rho_c(x_{2k+1}, x_{2k+2}) + \beta \rho_c(x_{2k+2}, x_{2k+3}) + \gamma \rho_c(x_{2k+2}, x_{2k+3}) \]

\[ \rho_c(x_{2k+2}, x_{2k+3}) \lesssim \frac{\alpha}{1 - \beta - \gamma} \rho_c(x_{2k}, x_{2k+1}) \rho_c(x_{2k+1}, x_{2k+2}). \]

Assume, \( h = \frac{\alpha}{1 - \beta - \gamma} < 1 \), we have

\[ \rho_c(x_{n+1}, x_{n+2}) \lesssim hd(x_n, x_{n+1}) \lesssim ... \lesssim h^{n+1} \rho_c(x_0, x_1). \]

For some \( m > n \), we have

\[ \rho_c(x_n, x_m) \lesssim \rho_c(x_n, x_{n+1}) + \rho_c(x_{n+1}, x_{n+2}) + ... + \rho_c(x_{m-1}, x_m) \]

\[ \lesssim [h^n + h^{n+1} + ... + h^{m-1}] \rho_c(x_0, x_1) \]

\[ \lesssim \left[ \frac{h^n}{1 - h} \right] \rho_c(x_0, x_1). \]

This implies,

\[ |\rho_c(x_m, x_n)| \leq \left[ \frac{h^n}{1 - h} \right] |\rho_c(x_0, x_1)| \to 0, \text{ as } m, n \to \infty. \]
Hence, \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete complex valued metric space, therefore there exists \( u \in X \) such that \( x_n \to u \), we shall show that \( u = Su \). To prove, consider \( \rho_c(u, Su) = z > 0 \). Therefore, by using triangle inequality, we have

\[
\rho_c(u, Su) = z \lesssim \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+2}, Su)
\]

\[
\lesssim \rho_c(u, x_{2k+2}) + \rho_c(Tx_{2k+1}, Su)
\]

\[
\lesssim \rho_c(u, x_{2k+2}) + \alpha \rho_c(x_{2k+1}, u) + \beta \frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u)}
\]

\[
+ \gamma \frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(u, Su) + \rho_c(x_{2k+1}, Tx_{2k+1})}
\]

\[
\lesssim \rho_c(u, x_{2k+2}) + \alpha \rho_c(x_{2k+1}, u) + \beta \frac{z \rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u)}
\]

\[
+ \gamma \frac{z \rho_c(x_{2k+1}, x_{2k+2})}{1 + z + \rho_c(x_{2k+1}, x_{2k+2})}.
\]

This implies,

\[
|\rho_c(u, Su)| \lesssim |\rho_c(u, x_{2k+2})| + \alpha |\rho_c(x_{2k+1}, u)| + \beta \frac{|z| |\rho_c(x_{2k+1}, x_{2k+2})|}{1 + \rho_c(x_{2k+1}, u)}
\]

\[
+ \gamma \frac{|z| |\rho_c(x_{2k+1}, x_{2k+2})|}{1 + z + \rho_c(x_{2k+1}, x_{2k+2})}.
\]

Letting \( k \to \infty \), we have \( |\rho_c(u, Su)| \leq 0 \), hence \( \rho_c(u, Su) = 0 \). That is \( z = 0 \), a contradiction. Hence our supposition is wrong. Therefore, \( z = 0 \), ie \( Su = u \). On the same lines, we can show that \( u = Tu \). Therefore \( u \) is a common fixed point of \( S \) and \( T \).

Now, we shall show that \( u \) is a unique common fixed point of \( S \) and \( T \). For this, Consider \( u^* = u \) be another common fixed point of \( S \) and \( T \). Therefore,

\[
\rho_c(u, u^*) = \rho_c(Su, Tu^*)
\]

\[
\lesssim \alpha \rho_c(u, u^*) + \beta \frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, u^*)} + \gamma \frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, Su) + \rho_c(u, Tu^*)}
\]

\[
\lesssim \alpha \rho_c(u, u^*).
\]

This implies \( (1 - \alpha)\rho_c(u, u^*) \leq 0 \) and hence, \( (1 - \alpha)|\rho_c(u, u^*)| \leq 0 \). Therefore, \( \rho_c(u, u^*) = 0 \) and hence, \( u = u^* \), which implies uniqueness. Thus, \( u \) is a unique common fixed point of \( S \) and \( T \).

**Corollary 3.3.** Let \((X, \rho_c)\) be a complete complex valued metric space and \( T : X \to X \) be a self mapping satisfying the following condition:

\[
\rho_c(Tx, Ty) \lesssim \alpha \rho_c(x, y) + \beta \frac{\rho_c(x, Tx)\rho_c(y, Ty)}{1 + \rho_c(x, y)} + \gamma \frac{\rho_c(x, Tx)\rho_c(y, Ty)}{1 + \rho_c(x, Tx) + \rho_c(y, Ty)}
\]
for all \( x, y \in X \), where \( \alpha, \beta, \gamma \) are non-negative reals with \( \alpha + \beta + \gamma < 1 \). Then \( T \) has a unique fixed point.

**Corollary 3.4.** Let \( (X, \rho_c) \) be a complete complex valued metric space and \( T : X \to X \) be a self mapping satisfying the following condition:

\[
\rho_c(T^nx, T^ny) \preceq \alpha \rho_c(x, y) + \beta \frac{\rho_c(T^n x, T^n y) \rho_c(x, y)}{1 + \rho_c(x, y)} + \gamma \frac{\rho_c(x, T^n x) \rho_c(y, T^n y)}{1 + \rho_c(x, T^n x) + \rho_c(y, T^n y)}
\]

for all \( x, y \in X \), where \( \alpha, \beta, \gamma \) are non-negative reals with \( \alpha + \beta + \gamma < 1 \). Then \( T \) has a unique fixed point.

**Proof.** By Corollary 3.3, we obtain \( \eta \in X \) such that \( T^n \eta = \eta \).

The result then follows from the fact that,

\[
\rho_c(T^n \eta, \eta) = \rho_c(TT^n \eta, T^n \eta) = \rho_c(T^n T \eta, T^n \eta)
\]

\[
\preceq \alpha \rho_c(T \eta, \eta) + \beta \frac{\rho_c(T \eta, T^n T \eta) d(\eta, T^n \eta)}{1 + \rho_c(T \eta, T^n \eta) + d(\eta, T^n \eta)} + \gamma \frac{\rho_c(T \eta, T^n \eta) d(\eta, T \eta)}{1 + \rho_c(T \eta, T^n \eta) + d(\eta, T \eta)}
\]

\[
= \alpha \rho_c(T \eta, \eta).
\]

Therefore, \( (1 - \alpha) \rho_c(T \eta, \eta) \preceq 0 \), this implies, \( (1 - \alpha)|\rho_c(T \eta, \eta)| \leq 0 \), hence \( \rho_c(T^n \eta, \eta) = 0 \).

Thus, \( \eta \) is a fixed point of \( T \). On the same lines of Theorem 3.2, we can prove the uniqueness.

4. Deduction

**Theorem 4.1.** [Azam et al.] Let \( (X, \rho_c) \) be a complete complex valued metric space and let the mappings \( S, T : X \to X \) satisfy:

\[
\rho_c(Sx, Ty) \preceq \lambda \rho_c(x, y) + \mu \frac{\rho_c(Sx, Sx) \rho_c(y, Ty)}{1 + \rho_c(x, y)}
\]

for all \( x, y \in X \), where \( \lambda, \mu \) are non-negative reals with \( \lambda + \mu < 1 \). Then \( S, T \) have a unique common fixed point.

**Proof.** The required result can be obtained by assuming \( \gamma = 0 \) in Theorem 3.1 and 3.2.

**References**


