SOME REMARKS ON THE GENERALIZED MYERS THEOREMS

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Abstract. In this paper, firstly, we prove a generalization of Ambrose (or Myers) theorem for the Bakry-Emery Ricci tensor. Later, we improve the diameter estimate obtained by Galloway for complete Riemannian manifolds. To obtain these results, we utilize the Riccati inequality and the index form of a minimizing unit speed geodesic segment, respectively.

1. Introduction

Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\) and let \(f\) be a smooth function on \(M\). By the Bakry-Emery Ricci tensor we mean

\[
\text{Ric}_f := \text{Ric} + \text{Hess}f,
\]

where \(\text{Ric}\) and \(\text{Hess}f\) are the Ricci tensor and the Hessian of \(f\), respectively [2].

When \(f\) is a constant function, the Bakry-Emery Ricci tensor becomes the original Ricci tensor. We recall Ambrose’s result [1], which gives an important generalization of the Myers compactness theorem [13] for the original Ricci tensor as another variant.

Theorem 1.1. [1] If there exists a point \(p \in M\) such that the condition

\[
\int_0^\infty \text{Ric}(\gamma'(t), \gamma'(t))dt = \infty
\]

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holds along every geodesic $\gamma(t)$ emanating from $p \in M$, then manifold is compact.

In [19], Zhang proved the Ambrose’s compactness theorem for the Bakry-Emery Ricci tensor given in (1.1).

**Theorem 1.2.** [19] If there exists a point $p \in M$ such that every geodesic $\gamma(t)$ emanating from $p$ satisfies

$$\int_0^\infty \text{Ric}_f(\gamma'(t), \gamma'(t))dt = \infty, \quad (1.3)$$

and $f(x) \leq C(d(x, p) + 1)$ for some constant $C$, where $d(x, p)$ is the distance from $p$ to $x$, then $M$ is compact.

Another generalization has been considered by Cavalcante-Oliveira-Santos in [3], where the condition on $f$ given in Theorem 1.2 is replaced with a condition on the derivation of $f$ as follows:

**Theorem 1.3.** [3] Suppose that there exists a point $p$ in a complete manifold $M$ such that every geodesic $\gamma(t)$ emanating from $p$ satisfies

$$\int_0^\infty \text{Ric}_f(\gamma'(t), \gamma'(t))dt = \infty, \quad (1.4)$$

and $\frac{df}{dt} \leq 0$. Then $M$ is compact.

The proofs of the above theorems are based on the Riccati inequality and a careful analysis of this inequality being different from calculus of variations. Moreover, these theorems do not require that the original Ricci tensor and the Bakry-Emery Ricci tensor be everywhere non-negative. However, these results cannot give an upper bound for the diameter of a manifold.

Our first aim is to improve condition on the function $f$ under the same $\text{Ric}_f$ assumption as in the Theorem 1.3.

On the other hand, Galloway [6] proved a perturbed version of Myers compactness theorem by the derivative in the radial direction of some bounded function as follows:

**Theorem 1.4 (Galloway).** Let $M$ be a complete Riemannian manifold and $\gamma$ be a geodesic joining two points of $M$. Suppose that

$$\text{Ric}(\gamma'(t), \gamma'(t)) \geq a + \frac{d\phi}{dt} \quad (1.5)$$
holds along $\gamma$ for some constant $a > 0$, and $| \phi | \leq c$ for some constant $c \geq 0$. Then $M$ is compact and
\[
\text{diam}(M) \leq \frac{\pi}{a} (c + \sqrt{c^2 + a(n-1)}).
\] (1.6)

Our second aim is to show that there is a sharper diameter estimate than Galloway's diameter estimate (1.6).

We are now ready to give our main theorems.

**Theorem 1.5.** Let $(M, g)$ be a complete Riemannian manifold of dimension $n \geq 2$. Suppose there exists a point $p \in M$ such that every geodesic $\gamma(t)$ emanating from $p$ satisfies
\[
\int_0^\infty \text{Ric}_f(\gamma'(t), \gamma'(t)) \, dt = \infty,
\] (1.7)
and $f'(t) \leq \frac{1}{4}(1 - \frac{1}{t})$ for all $t \geq 1$, then manifold is compact.

In the above theorem, we provide that the condition $f'(t) \leq 0$ given in Theorem 1.3 for $t = 1$. In order to prove Theorem 1.5, we use the Riccati inequality.

**Theorem 1.6.** Let $(M, g)$ be a complete Riemannian manifold and $\gamma$ be a geodesic joining two points of $M$. Suppose that
\[
\text{Ric}(\gamma'(t), \gamma'(t)) \geq a + \frac{d\phi}{dt}
\] (1.8)
holds along $\gamma$ for some constant $a > 0$, and $| \phi | \leq c$ for some constant $c \geq 0$. Then $M$ is compact and
\[
\text{diam}(M) \leq \frac{1}{a} \left( 2c + \sqrt{4c^2 + a(n-1)} \pi^2 \right).
\] (1.9)

The diameter estimate (1.9) above is sharper than (1.6) by Galloway. In order to prove above theorem, we use the index form of a minimizing unit speed geodesic segment. For basic facts about this topic, we refer the reader to the book [8,14].

**Remark 1.1.** There exists many varied examples of compactness theorems involving the original Ricci tensor and modified Ricci tensors; see for instance [4,5,7,9–12,15–18].

2. Proofs of the Theorems

Before stating our main results, we recall the definitions of gradient, Hessian and Laplacian of any smooth function $f \in C^\infty(M)$ on a Riemannian manifold. The gradient, Hessian and Laplacian are defined by
\[
g(\nabla f, V) = V(f), \quad (\text{Hess}(f))(V, W) = g(\nabla_V \nabla f, W) \quad \text{and} \quad \Delta f = \text{tr}(\nabla^2 f)
\] (2.10)
for all vector fields $V, W$, respectively. The Riemannian curvature tensor is defined as

$$R(V, W)Z = \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V, W]} Z,$$

(2.11)

and the Ricci curvature as

$$\text{Ric}(V, W) = \sum_{i=1}^{n} g(R(E_i, V) W, E_i)$$

(2.12)

for all vector fields $V, W, Z$, where $\{E_i\}_{i=1}^{n}$ is an orthonormal frame of $(M, g)$ Riemannian manifold.

**Proof of Theorem 1.5** We assume that $M$ is a non-compact Riemannian manifold and let $\gamma(t)$ be an unit speed ray starting from $p$. For every $t > 0$, $m(t)$ denotes the Laplacian of distance function from a fixed point $p \in M$. We know from some calculations with the Bochner formula that this gives the following Riccati inequality

$$m'(t) + \frac{1}{n-1} m^2(t) + \text{Ric}(\gamma'(t), \gamma'(t)) \leq 0.$$  

(2.13)

We consider a smooth function $F(t)$ defined by

$$F(t) := m(t) + \zeta(t)$$

(2.14)

for all $t > 0$, where $\zeta \in \mathcal{C}^\infty(M)$. The derivation of $F(t)$ gets

$$F'(t) = m'(t) + \zeta'(t).$$

(2.15)

Combining (2.13) and (2.15), we obtain

$$F'(t) - \zeta'(t) + \frac{1}{n-1} m^2(t) + \text{Ric}(\gamma'(t), \gamma'(t)) \leq 0.$$  

(2.16)

It is clear that we have

$$m(t) = F(t) - \zeta(t),$$

(2.17)

by (2.14). Substituting (2.17) into (2.16), we obtain

$$F'(t) - \zeta'(t) + \frac{1}{n-1} (F(t) - \zeta(t))^2 + \text{Ric}(\gamma'(t), \gamma'(t)) \leq 0.$$  

(2.18)

Using the essential inequality $(x + y)^2 \geq \frac{1}{\alpha+1} x^2 - \frac{1}{\alpha} y^2$ holding for all real numbers $x, y$ and positive real number $\alpha$, we get

$$\left( F(t) - \zeta(t) \right)^2 \geq \frac{1}{\alpha+1} F^2(t) - \frac{1}{\alpha} \zeta^2(t).$$  

(2.19)
Substituting \( (2.19) \) into \( (2.18) \) and taking \( \alpha = \frac{1}{n-1} > 0 \), we have
\[
\text{Ric}(\gamma'(t), \gamma'(t)) \leq -F'(t) + \zeta'(t) - \frac{1}{n} F^2(t) + \zeta^2(t).
\] (2.20)

If we add \((\text{Hess} f)(\gamma'(t), \gamma'(t))\) to the both sides of inequality \( (2.20) \), we have
\[
\text{Ric}_f(\gamma'(t), \gamma'(t)) \leq -F'(t) + \zeta'(t) - \frac{1}{n} F^2(t) + \zeta^2(t) + (\text{Hess} f)(\gamma'(t), \gamma'(t)).
\] (2.21)

Integrating both sides of the inequality \( (2.21) \) from 1 to \( t \), we obtain
\[
\int_1^t \text{Ric}_f(\gamma'(s), \gamma'(s)) ds \leq -F(t) + F(1) - \int_1^t \frac{1}{n} F^2(s) ds + \int_1^t (\zeta'(s) + \zeta^2(s)) ds
\]
\[
+ g(\nabla f, \gamma')(t) - g(\nabla f, \gamma')(1).
\] (2.22)

Therefore, under the assumption
\[
\int_0^\infty \text{Ric}_f(\gamma'(t), \gamma'(t)) dt = \infty
\] (2.23)
given in Theorem 1.5, we have
\[
\lim_{t \to \infty} -F(t) - \int_1^t \frac{1}{n} F^2(s) ds + \int_1^t (\zeta'(s) + \zeta^2(s)) ds + f'(t) = \infty,
\] (2.24)
where \( f' = \frac{d}{dt} F(\gamma(t)) = g(\nabla f, \gamma') \). Here, multiplying by \( 1/n \) on both sides then yields
\[
\lim_{t \to \infty} \frac{1}{n} F(t) - \frac{1}{n} \int_1^t \left( \frac{1}{n} F(s) \right)^2 ds + \frac{1}{n} \int_1^t (\zeta'(s) + \zeta^2(s)) ds + \frac{1}{n} f'(t) = \infty.
\] (2.25)

Because of \( (2.24) \), given \( C > 1 \) there exists \( t_1 > 1 \) such that
\[
- \frac{1}{n} F(t) - \frac{1}{n} \int_1^t \left( \frac{1}{n} F(s) \right)^2 ds + \frac{1}{n} \int_1^t (\zeta'(s) + \zeta^2(s)) ds + \frac{1}{n} f'(t) \geq C
\] (2.26)
for all \( t \geq t_1 \).

On the other hand, under the assumption \( f'(t) \leq \frac{1}{4}(1 - \frac{1}{t}) \) of Theorem 1.5, if the function \( \zeta \) is taken to be \( \zeta(t) = \frac{1}{2t} \), then we get the following inequality
\[
- \frac{1}{n} F(t) - \frac{1}{n} \int_1^t \left( \frac{1}{n} F(s) \right)^2 ds \geq C
\] (2.27)
for all \( t \geq t_1 \).

Let us now consider an increasing sequence \( \{t_\ell\} \) defined by
\[
t_{\ell+1} = t_\ell + C^{1-\ell}, \quad \text{for} \quad \ell \geq 1,
\] (2.28)
such that \( \{t_\ell\} \) converges to \( T := t_1 + \frac{C}{\ell-1} \) as \( \ell \to \infty \).

We claim the fact that \( -F(t) \geq nC^\ell \) for all \( t \geq t_\ell \): To prove the claim, we use induction argument. It is trivial from inequality \( (2.27) \) for \( \ell = 1 \). By induction, we get the claim for \( \ell \).
Then we must prove that $-F(t) \geq nC^{t+1}$ for all $t \geq t_{\ell+1}$. By means of the inequality (2.27), we obtain

$$
-F(t) \geq nC + \frac{1}{n} \int_{1}^{t} F^2(s)ds \\
\geq \frac{1}{n} \int_{t_{\ell}}^{t} F^2(s)ds + \frac{1}{n} \int_{t_{\ell}}^{t} F^2(s)ds \\
\geq \frac{1}{n} \int_{t_{\ell}}^{t} F^2(s)ds \\
\geq nC^{2\ell}(t - t_{\ell}) \\
\geq nC^{2\ell}(t_{\ell+1} - t_{\ell}) = nC^{\ell+1}.
$$

(2.29)

This proves the above claim.

From hence, we have

$$
\lim_{\ell \to \infty} -F(t_{\ell}) = -F(T) \geq \lim_{\ell \to \infty} nC^{\ell}.
$$

(2.30)

However, this result contradicts with the smoothness of $F(t)$. Namely, $\lim_{t \to T^{-}} -F(t) = \infty$. This completes the proof of Theorem 1.5.

On the other hand, under the same assumptions given in the Theorem 1.4, we see that, the above diameter estimate given by (1.6) can be improved as follows:

**Proof of Theorem 1.6** Let $p, q \in M$ be two distinct point and $\gamma$ a minimizing unit speed geodesic segment from $p$ to $q$ of length $\ell > 0$. Let $\{E_1 = \gamma', E_2, \ldots, E_n\}$ be a parallel orthonormal frame along $\gamma$ and let $h \in C^\infty([0, \ell])$ be a real-valued smooth function such that $h(0) = h(\ell) = 0$. Then, from the index form of $\gamma$, we have

$$
\sum_{i=2}^{n} I(hE_i, hE_i) = \int_{0}^{\ell} \left( (n-1)h'^2 - h^2 \text{Ric}(\gamma', \gamma') \right)dt.
$$

(2.31)

Using the assumption (1.8) given in Theorem 1.6 in the integral expression (2.31), we get

$$
\sum_{i=2}^{n} I(hE_i, hE_i) \leq \int_{0}^{\ell} \left( (n-1)h'^2 - ah^2 - h^2 \frac{d\phi}{dt} \right)dt.
$$

(2.32)

In the inequality (2.32), the term $-h^2 \frac{d\phi}{dt}$ equals to

$$
-h^2 \frac{d\phi}{dt} = -\frac{d}{dt} (h^2 \phi) + 2hh' \phi.
$$

(2.33)

Integrating both sides of (2.33), we get

$$
\int_{0}^{\ell} -h^2 \frac{d\phi}{dt} dt = 2 \int_{0}^{\ell} hh' \phi dt \leq 2 \int_{0}^{\ell} |hh'| \phi |dt \leq 2c \int_{0}^{\ell} |hh'| dt.
$$

(2.34)
Thus, under the choice \( h(t) = \sin(\frac{\pi t}{\ell}) \), we have

\[
\sum_{i=2}^{n} I(hE_i, hE_i) \leq \frac{1}{2\ell} \left[ (n - 1)\pi^2 - a\ell^2 + 4c\ell \right].
\] (2.35)

Since \( \gamma \) is a minimal geodesic, we must take

\[
a\ell^2 - 4c\ell - (n - 1)\pi^2 \leq 0.
\] (2.36)

This inequality gives

\[
\ell \leq \frac{1}{a} \left( 2c + \sqrt{4c^2 + a(n - 1)\pi^2} \right).
\] (2.37)

This completes the proof of Theorem 1.6.

References

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