ON THE MEASURE OF TRANSCENDENCE OF $\zeta = \sum_{k=0}^{\infty} G_k^{-e_k}$ FORMAL LAURENT SERIES

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ABSTRACT. In this work, we determine the transcendence measure of the formal Laurent series, $\varsigma = \sum_{k=0}^{\infty} G_k^{-e_k}$ whose transcendence has been established by S. M. SPENCER [15]. Using the methods and lemmas in P. Bundschuh’s article measure of transcendence for the above $n$ is determined as

$$T(n, H) = H^{-(d+1)q^d - edq^{2d}}.$$ 

On the other hand, it was proven that transcendence series $\eta$ is not a $U$ but is a $S$ or $T$ numbers according to the Mahler’s classification.

1. INTRODUCTION

Let $p$ a prime number and $u \geq 1$ an integer. Let $F$ be a finite field with $q = p^u$elements. We denote the ring of the polynomials with in one variable over $F$ by $F[x]$ and its quotient field by $F(x)$. If $a \in F[x]$ is a non-zero polynomial, denote by $\partial a$ its degree. If $a = 0$, then its degree is defined as $\partial 0 := -\infty$. Let $a$ and $b$ ($b \neq 0$) two polynomials from $F[x]$ and define a discrete valuation of $F(x)$ as follows

$$\left| \frac{a}{b} \right| = q^{\partial a - \partial b}.$$
Let $K$ be the completion of $F(x)$ with respect to this valuation. Every element $\omega$ of $K$ can be uniquely represented by

$$\omega = \sum_{n=-k}^{\infty} c_n x^{-n}, \ c_n \in F.$$ 

If $\omega = 0$, then all $c_n$ are zero. If $\omega \neq 0$, then there exist and $k \in \mathbb{Z}$ for which $c_k \neq 0$. If $\omega \neq 0$, then we have $|\omega| = q^{-k}$.

Therefore $K$ is the field all Formal Laurent series. The classical theory of transcendence over complex numbers has a similar version over $K$. Elements of $F[x]$ and $F(x)$ correspond to integers and fractions of the classical theory, respectively.

If $\omega$ is one of the roots of a non-zero polynomial with coefficients in $F[x]$, then $\omega \in K$ is said to be algebraic over $F(x)$. Otherwise, $\omega$ is called transcendental over $F(x)$.

The studies to find transcendental numbers in $K$ were initiated first by Wade [16-19]. Also Geijsel [4-7] did similar studies. As it is the case in the classical theory of transcendental numbers, it is possible to define a measure of transcendence.

The measure of transcendence is thoroughly studied in the classical theory. For example, the transcendence measure of $e$ has been widely investigated by Mahler [9], Fel’dman [3] and Cijsw [2]. Example for transcendence measure in the field $K$ have been given for the first time by Bundschuh [1]. Further more, Özdemir showed the measure of transcendence of some Formal Laurent series [11],[12].

In this work, we determine the transcendence measure of some Formal Laurent series whose transcendence has been established by S.M.Spencer [15]. We take the $G_0 | G_1 | G_2, \ldots, d \in g G_0 \geq 1, e = e_0 < e_1 < e_2 < \ldots, < e_k | e_{k+1}^{e_1} / e_2 \neq p^r \text{ for } r > s, e_k \in \mathbb{Z}.$

If $G \in F[x]$ is a fixed non-zero polynomial of degree, $\partial(G_k) = g_k, g \geq 1$ then the series

$$\varsigma = \sum_{k=0}^{\infty} G_k^{-e_k}$$  \hspace{1cm} (1)

is an element of $K$,and S.M.Spencer showed its transcendence in [14].

Using the methods and lemmas in Bundschuh’s article [1], we determine a transcendence measure of $\varsigma$. We take and arbitrary non-zero polynomial

$$P(y) = \sum_{v=0}^{n} a_v y^v, \ (a_v \in F[x]; v = 0, 1, \ldots, n)$$  \hspace{1cm} (2)

Whose degree $\partial(P)$ is less than or equal to $n$. The height of $P$ is denoted by

$$h(p) = \max_{v=0}^{\infty} |a_v| = q^{\max_{v=0}^{\infty} \partial(a_v)}$$

For the transcendental element $\varsigma = \sum_{k=0}^{\infty} G_k^{-e_k}$ of $K$, we define the positive quantity

$$\Lambda_n(H, \varsigma) = \min |P(\varsigma)|,$$
where \( P \neq 0, \partial(P) \leq n, h(P) \leq H \). If \( T(n, H) \) is a function of the variables \( n, H \) of \( \Lambda_n(H, \varsigma) \) which satisfies the inequality

\[
\Lambda_n(H, \varsigma) \geq T(n, H)
\]

for all sufficiently large values of \( n \) and \( H \), then \( T(n, H) \) is said to be a transcendence measure of \( \varsigma \).

2. Preliminaries

**Theorem 2.1.** We take an arbitrary, non-zero polynomial

\[
P(y) = \sum_{v=0}^{n} a_v y^v, (a_v \in F[x]; v = 0, 1, \ldots, n)
\]

Further let \( \partial(P) = d, h(p) = h \) and \( a = \max_{v=0}^{d} \partial a_v \).

\[
d p^m \log h \geq g_k e_k \log q.
\]

Then we have

\[
|P(\xi)| \geq h^{-(d+1)q^d - \epsilon d q^{2d}}
\]

and the transcendence measure of \( \omega \) is

\[
T(n, H) = H^{-(d+1)q^d - \epsilon d q^{2d}}
\]

As in the classical theory of transcendental number theory (see Schneider [13], Pag. 6), it is possible to define Mahler’s classification on \( K \). Let \( K \) be transcendental, and define :

\[
\Theta_n(H, \eta) := \lim_{H \to \infty} \sup_{n} \frac{-\log \Theta_n(H, \eta)}{\log H}
\]

\[
\Theta(\eta) := \lim_{n \to \infty} \sup \frac{1}{n} \Theta_n(\eta)
\]

Hence \( \Theta_n(\eta) \geq n \) for every \( n \in N \) and so \( \Theta(\eta) \geq 1 \). For every \( n, H \in N \),

\[
\Theta_n(H, \eta) < H^{-n} q^n \max(1, |\eta|^n)
\]

is satisfied (see Bundschuh [1], Lemma 3).

On the other hand, let the least natural number \( n \) satisfying \( \Theta_n(\eta) \geq \infty \) be donated by \( \mu(\eta) \). If there is no such \( n \), then on may define \( \mu(\eta) \) as \( \infty \). In this case, the transcendental number \( \eta \in R \) is called

- S-Laurent series if \( 1 \leq \Theta(\eta) < \infty \) and \( \mu(\eta) = \infty \),
- T-Laurent series if \( \Theta(\eta) = \infty \) and \( \mu(\eta) = \infty \),
- U-Laurent series if \( \Theta(\eta) = \infty \) and \( \mu(\eta) < \infty \).
Moreover the $U$-class may be divided into subclasses. If $\mu(\eta) = m$ ($m > 0$), then $\eta$ is called a $U_m$-Laurent series. Le Vaque [8] was the first to show that for all $m$, $U_m$ is non-empty in the classical theory but the honour goes to Oryan [10] if the ground field is $K$.

According to the above classification, the series defined in (1) can not be a $U$-Laurent series. This fact may be proved by the help of the Theorem 2.1.

**Theorem 2.2.** The $\eta$ Laurent series defined by (1) doesn’t belong to the class $U$ so that it belongs to the class $S$ or to the class $T$.

We will use the following lemmas in proof of the theorem.

**Lemma 2.1.** Let

$$P(y) = \sum_{v=0}^{n} a_v y^v$$

$a_v \in F[x], \ a_d \neq 0 \ (d \geq 1), \ a = \max \partial a_v$ \ (10)

Then there are some elements $A_0, A_1, \ldots, A_d \in F[x]$, not all zero satisfying.

$$\partial A_1 \leq a_d (q^d - d + 1) \ \text{for} \ 0 \leq j \leq d \ \text{and}$$

$$\sum_{j=0}^{d} A_j y^{q^j} = p(y) \sum_{j=0, q^j \geq d}^{d} A_j \sum_{k=0}^{q^j - d} b_k a_d^{-k-1} y^{q^j - d - k} =: P(y)Q(y)$$ \ (11)

where $b_0 := 1$ and $b_k$, for $k \geq 1$ is the sum of product of exactly $k$ terms from $a_0, a_1, \ldots, a_d$, multiplied by $(\pm)$.

**Proof.** See the [1], lemma 4, page 416.

**Lemma 2.2.** Let $\eta \in K$ and $|\eta| = q^\lambda$. Under the hypotheses of Lemma 1 we have

$$|Q(\eta)| \leq q^{a_d(q^d - d + 1) + (q^d - d) \max(a, \lambda)}.$$ \ (12)

**Proof.** See the [1], lemma 5, page 417.

3. Proof of the Theorems

**Proof.** (Theorem 1)

Consider the polynomial defined by (4). With $\partial(p) = d, a_d \neq 0$. The Theorem is true obliviously for $d = 0$. Because then $|P(\eta)| = |a_0|$. $a_0 \in F[x]$ and since $a_0 \neq 0$ and we have, $|a_0| = q^{\beta(a_0)} > 1$. So the left side of (6) is less then 1. Let $d \geq 1$. By Lemma 1 there are some elements the $A_0, A_1, \ldots, A_d \in F[x]$ not all zero, such that

$$\sum_{j=0}^{d} A_j y^{q^j} = p(y) \sum_{j=0, q^j \geq d}^{d} A_j \sum_{k=0}^{q^j - d} b_k a_d^{-k-1} y^{q^j - d - k} =: P(y)Q(y)$$ \ (13)

$$\partial A_j \leq a_d (q^d - d + 1) \leq a_d q^d (0 \leq j \leq d)$$ \ (14)
In (13) we put $\eta$ instead of $y$ and using the fact that $F$ is a field having $q$ elements. We get

$$P(\eta)Q(\eta) = \sum_{j=0}^{d} A_j \eta^q$$

Separate the above sum as $S_1 + S_2$, where

$$S_1 = G^{e_\beta q^d} \sum_{j=0}^{d} A_j \sum_{k=0}^{k_j} G^{-e_k q^d} \quad \text{and} \quad S_2 = G^{e_\beta q^d} \sum_{j=0}^{d} A_j \sum_{k=k_j+1}^{\infty} G^{-e_k q^d}$$

(16)

where $\beta$ is non-negative integer to be chosen later. Let the rational integers $k_j (j = 0, 1, ..., d)$ be defined by

$$q^{j-d} e_{kj} < e_\beta \leq q^{j-d} e_{kj+1}$$

(17)

1) First, we prove that $|S_1| \geq 1$. That is, we prove $S_1$ is a polynomial but not equal zero. Their terms of the $S_1$ are

$$G^{e_\beta q^d} A_j G^{-e_k q^d} = A_j G^{e_\beta q^d - e_k q^d}$$

We show that $e_\beta q^d - e_k q^d \geq 0$ by (17), and since $k$ ranges from 0 to $k_j$ in the sum $S_1$. We have

$$e_\beta q^d - e_k q^d \geq q^j (e_{kj} - e_{kj}) \geq 0$$

(20)

which implies (19). By (19) and (18), $S_1$ is polynomial. Now we show $S_1$ isn’t identically zero as equivalently. We have equality in (19) when and only when $k = \beta$ and $j = d$. If we write the terms of $S_1$, we find

$$S_1 = A_0 \left( \sum_{k=0}^{k_0} G^{e_\beta q^d - e_k q^d} \right) + ... + A_d \left( \sum_{k=0}^{k_d} G^{e_\beta q^d - e_k q^d} \right)$$

(18)

$$S_1 = A_0 \left( G^{e_\beta q^d - e_0 q^d} + ... + G^{e_\beta q^d - e_{k_0} q^d} \right) + ... + A_d \left( G^{e_\beta q^d - e_0 q^d} + ... + G^{e_\beta q^d - e_{k_d} q^d} \right)$$

(21)

$$\mu := \min_{j=0}^{d-1} (e_\beta q^d - e_{kj} q^d, e_\beta q^d - e_{\beta-1} q^d)$$

(22)

$G^\mu$ divides of all terms in the sum(21) except only one term. Therefore,

$$S_1 = G^\mu R + A_d \quad (R \in F[x])$$

(23)

and hence we find

$$S_1 \equiv A_d \pmod{G^\mu}$$

(24)

Since $h = h(P) = q^a$,

$$a = \frac{\log h}{\log q}$$

(25)

By (5) and (25) we find

$$ad q^d \geq \frac{q}{e}$$

(26)

From (19) and (26) it holds (27). For this. Consider the sequence

$$\{e_{-1}, e = e_0, e_1, e_2, ...\}.$$

There are $\beta$ non-negative integers such that

$$e_{\beta-1} \leq \frac{ad q^d}{g} < e_\beta$$

(27)
From (27) we obtain the following statement for the above $\beta$

\[
\frac{adq^d}{g} < e_\beta \leq \frac{eadq}{g}
\]  

(28)

By (17) we have $e_\beta q^{d-j} \geq e_{k_j} \implies q^{d-j} \geq \frac{e_{k_j}}{e_\beta} \implies q^{d-j} - \frac{e_{k_j}}{e_\beta} \geq 0$. Hence we obtain

\[
q^{d-j} - \frac{e_{k_j}}{e_\beta} \geq 1\quad (j < d)
\]  

(29)

further, since $e_{\beta-1} < e_\beta \implies \frac{e_{\beta-1}}{e_\beta} < 1 \implies 0 < 1 - \frac{e_{\beta-1}}{e_\beta}$. Thus we get

\[
1 - \frac{e_{\beta-1}}{e_\beta} \geq 1
\]  

(30)

From (22),

\[
\mu = e_\beta \min_{j=0}^{d-1} q^j \left( q^{d-j} - \frac{e_{k_j}}{e_\beta} \right) q^d \left( 1 - \frac{e_{\beta-1}}{e_\beta} \right)
\]  

(31)

by (29), (30) and (31) and $q^{d_j}q^j > 1$ we get

\[
\mu > e_\beta
\]  

(32)

by (14), (28) and (32) we obtain

\[
g\mu > ge_\beta > adq^d > adq^d - d + 1 \geq \partial (A_d)
\]

that is,

\[
g\mu > \partial (A_d).
\]

this inequality means

\[
\partial (G^\mu) = g\mu > \partial (A_d).
\]

Hence we see $G^\mu$ doesn’t divide $A_d$. That is

\[
A_d \not\equiv 0 \pmod{G^\mu},
\]

by (28) and (36)

\[
S_1 \equiv A_d \not\equiv 0 \pmod{G^\mu}
\]  

(33)

therefore $S_1$ is not identically 0. so $S_1$ is a non-zero polynomial. so it is shown that $|S_1| \geq 1$.

2) we will show $|S_2| < 1$ since $k \geq k_j + 1$ in $S_2$, for the degree of the terms of $S_2$, we may write the following inequality from (14):

\[
\partial \left( G^{e_\beta q^d} A_j G^{-e_k q^d} \right) = \partial A_j + \partial G^{e_\beta q^d - e_k q^d}
\]

\[
\leq adq^d + g \left( e_\beta q^d - e_k q^d \right)
\]

\[
\leq adq^d + g \left( e_\beta q^d - e_{k_j+1} q^d \right)
\]

\[
\leq adq^d - ge_\beta \left( \frac{e_{k_j+1}}{e_\beta} q^d - q^d \right)
\]  

(34)

by (17) $q^d e_\beta < q^d e_{k_j+1} \quad 0 < \frac{e_{k_j+1}}{e_\beta} q^d - q^d$ is an integer. further, by (27) we obtain

\[
adq^d < ge_\beta
\]  

(35)
from (34), (35) and since \(\frac{e_{k+1}}{e_k}q^j - q^d\) is positive integer, we get

\[
\partial \left( G^e \partial_j G^{-e_k} q^j \right) < 0
\]

that is, the terms of \(S_2\) have negative degrees. this means

\[
|S_2| < 1
\]

3) we will prove the claim of the theorem. by the definition of \(S_1\) and \(S_2\), we can write \(S_1 + S_2 = G^e \partial_j P(\eta) Q(\eta)\). hence we obtain

\[
|S_1 + S_2| = \left| G^e \partial_j \right| |P(\eta)||Q(\eta)|
\]

(36)

since \(|S_1| \geq 1\) and \(|S_2| < 1\), we get

\[
|S_1 + S_2| = \max(|S_1|,|S_2|) = |S_1|
\]

(37)

By (36) and (37), we obtain

\[
|P(\eta)||Q(\eta)| = |S_1| \left| G^e \partial_j \right|^{-1}
\]

(38)

let \(|\eta| = q^\lambda\). By (1) and since \(|G^e\partial_k| = q^{d\deg G^e_k} = q^{ge_k}\),

we get \(|\eta| = q^{-ge_0} = q^{-ge}\) therefore \(\lambda = -ge\). since \(\max(a, \lambda) = \max(a, -ge) = a\) and by lemma 2, we find

\[
|Q(\eta)| \leq q^{ad(q^d-d+1)+(q^d-d)\max(a, \lambda)} \leq q^{adq^d+aq^d} \leq q^{a(d+1)q^d}
\]

(39)

further, by (28)

\[
\left| G^e \partial_j \right| = q^{ge \partial_j q^d}
\]

\[
\leq q^{cadq^d q^d}
\]

\[
= q^{cadq^{2d}}
\]

(40)

by (38),(39),(40) and since \(|S_1| \geq 1\)

\[
|P(\eta)| = |S_1| \left| G^e \partial_j \right|^{-1} |Q(\eta)|^{-1}
\]

\[
\geq \left| G^e \partial_j \right|^{-1} |Q(\eta)|^{-1}
\]

\[
\geq q^{cadq^{2d} - a(d+1)q^d}
\]

(41)

by (41) and since \(h = q^a\)

\[
|P(\eta)| \geq h^{-(d+1)q^d - cadq^{2d}}
\]

this is the claim of the theorem 1.

**Proof.** (Theorem 2)

let the degree of the polynomial \(P\) in Theorem 1 be \(\partial(P) = d \leq n\) and let its height be

\[
h(P) = h \leq H \text{ by (6)},
\]

\[
|P(\eta)| \geq H^{-(n+1)q^n - \epsilon q^{2n}}.
\]

(42)
(42) and (5) and by the definition of Mahler’s classification

$$\Theta_n(H, \eta) \geq H^{-(n+1)q^n - cnq^{2n}}$$

for all sufficiently large natural numbers n and H. hence consequently

$$\log \Theta_n(H, \eta) \geq \left[ -(n + 1)q^n - cnq^{2n} \right] \log H$$

$$\frac{\log \Theta_n(H, \eta)}{\log H} \leq (n + 1)q^n - cnq^{2n}$$

(43)

$$\Theta_n(\eta) \geq \lim_{H \to \infty} \sup \frac{-\Theta_n(H, \eta)}{\log H} \leq enq^{2n} + (n + 1)q^n$$

(44)

that is, for every index n

$$\Theta_n(\eta) < \infty$$

by the definition of Mahler’s classification, $$\mu(\eta) = \infty$$. This shows $$\eta$$ can never to the class U so that it belongs to the class S or to class T.

**References**


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