ON SEMI-IN Variant ξ⊥-SUBMANIFOLDS OF LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. In the present paper, we study semi-invariant ξ⊥-submanifolds of Lorentzian para-Sasakian manifolds. We discuss the integrability conditions of the distributions $D$ and $D⊥$ on semi-invariant ξ⊥-submanifolds of Lorentzian para-Sasakian manifolds. We also obtain some characterizations for the totally umbilical semi-invariant ξ⊥-submanifolds of Lorentzian para-Sasakian manifolds.

1. Introduction


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of LP-Saskian manifold were studied by several geometers (see, [10], [11], [12], [13], [14]). N. Papaghiuc [15] defined $\xi^\perp$-submanifolds in which the structural vector field $\xi$ is orthogonal to the submanifolds and studied geometry of the leaves on Kenmotsu manifold. Constantin C. et. al [16] studied semi-invariant $\xi^\perp$-submanifolds of generalized quasi-Sasakian manifolds. M. M. Tripathi [17] studied semi-invariant $\xi^\perp$-submanifolds of trans-Sasakian manifold. Further, S.Y. Perktas et. al [18] studied semi-invariant $\xi^\perp$-submanifolds of P-Sasakian manifold. In this paper, we study semi-invariant $\xi^\perp$-submanifolds of LP-Sasakian manifold. In particular, we recover the results of Papaghiuc [15] and Calin [16].

The paper is organized as follows. In section 2, we give a brief description of Lorentzian para-Sasakian manifold. In section 3, we find some results on semi-invariant $\xi^\perp$-submanifolds of Lorentzian para-Sasakian manifolds, discuss the integrability of distributions $D$ and $D^\perp$ of semi-invariant $\xi^\perp$-submanifolds of Lorentzian para-Sasakian manifolds and finally in section 4, we find some characterizations for the totally umbilical semi-invariant $\xi^\perp$-submanifolds of Lorentzian para-Sasakian manifolds.

2. Preliminaries

Lorentzian para-Sasakian manifold

Let $\tilde{M}$ be $(2n + 1)$-dimensional almost contact metric manifold with a metric tensor $g$, a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1–form $\eta$ which satisfy

$$\phi^2 X = X + \eta(X)\xi, \eta(\xi) = -1, \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.2)$$

$$g(X, \xi) = \eta(X), \quad (2.3)$$

$$g(\phi X, Y) = g(X, \phi Y) \quad (2.4)$$

for all vector fields $X, Y$ tangent to $\tilde{M}$. Such a manifold is termed as Lorentzian para-contact manifold and the structure $(\phi, \eta, \xi, g)$ a Lorentzian para-contact structure [1]. Also in a Lorentzian para-contact structure the following relations hold:

$$\phi \xi = 0, \ \eta(\phi X) = 0, \ \text{rank}(\phi) = n - 1.$$ 

A Lorentzian para-contact manifold $\tilde{M}$ is called Lorentzian para-Sasakian (LP-Sasakian manifold if [2]).

$$\left(\tilde{\nabla}_X \phi\right)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (2.5)$$
for all vector fields $X$, $Y$ tangent to $\bar{M}$, where $\bar{\nabla}$ is the Riemannian connection with respect to $g$.

3. Semi-invariant $\xi^\perp$-submanifolds

Let $M$ be an $m$-dimensional submanifold of $\bar{M}$, isometrically immersed in $\bar{M}$. The tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as

$$T\bar{M} = TM \oplus T\bar{M}^\perp.$$ 

**Definition 3.1** [8] An $m$-dimensional Riemannian submanifold $M$ of a Lorentzian para-Sasakian manifold $\bar{M}$ is called a semi-invariant $\xi^\perp$-submanifold of Lorentzian para-Sasakian manifold if $\xi$ is normal to $M$ and there exists on $M$ a pair of distributions $(D, D^\perp)$ such that

(i) $TM$ orthogonally decomposes as $D \oplus D^\perp$,

(ii) the distribution $D_x$ is invariant under $\phi$, that is $\phi D_x \subseteq D_x$ for each $x \in M$,

(iii) the distribution $D^\perp$ is anti-invariant under $\phi$, that is $\phi D^\perp_x(M) \subseteq T^\perp_x(M)$ where $T_x M$ and $T^\perp_x M$ are tangent and normal spaces of $M$ at $x \in M$. If $D^\perp = 0$ then $M$ is an invariant $\xi^\perp$-submanifold. The normal bundle $T^\perp M$ can also be decomposed as

$$T^\perp M = \phi D^\perp \oplus \mu \oplus \{\xi\},$$

where $\phi \mu \subseteq \mu$.

Any vector $X$ tangent to $M$ is given by

$$X = PX + QX,$$  

where $PX$ and $QX$ belong to the distribution $D$ and $D^\perp$ respectively. Moreover, for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, we put

$$\phi X = tX + \omega X,$$  

where $tX$ (resp. $\omega X$) denotes the tangential (resp. normal) components of $\phi X$ and

$$\phi N = BN + CN,$$  

where $BN$ (resp. $CN$) denotes the tangential (resp. normal) component of $\phi N$. 

Gauss formula for semi-invariant $\xi^\perp$-submanifolds of an $LP-$ Sasakian manifold is given by

$$\nabla_X Y = \nabla_X Y + h(X,Y). \quad (3.4)$$

Weingarten formula is given by

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N \quad (3.5)$$

for any $X,Y \in TM$, $N \in T^\perp M$, where $h$ (resp. $A_N$) is the second fundamental form (resp. tensor) of $M$ in $\bar{M}$ and $\nabla^\perp$ denotes the operator of the normal connection. Moreover, we have

$$g(h(X,Y),N) = g(A_N X,Y). \quad (3.6)$$

Now, we study the integrability of both the distributions $D$ and $D^\perp$. For this purpose, first we establish some results for further use.

**Proposition 3.1.** Let $M$ be a semi-invariant $\xi^\perp$-submanifold of an $LP$-Sasakian manifold $\bar{M}$. Then

(a) $(\nabla_X t)Y = A_\omega Y X + Bh(X,Y), \quad (3.7)$

(b) $(\nabla_X \omega)Y = Ch(X,Y) - h(X,tY) + g(X,Y)\xi$

$\forall X,Y \in \Gamma(TM)$.

**Proof** In view of (3.2), (3.3), (3.4) and (3.5), we have

$$ (\bar{\nabla}_X \phi)Y = (\nabla_X t)Y - A_\omega Y X + (\nabla_X \omega)Y + h(X,tY) - \phi h(X,Y). \quad (3.8)$$

Using (2.6) in (3.8), we get

$$ g(X,Y)\xi + \phi h(X,Y) = (\nabla_X t)Y - A_\omega Y X + (\nabla_X \omega)Y + h(X,tY). \quad (3.9)$$

Comparing tangential and normal components of (3.9), we have our assertion.

We can state the following proposition.

**Proposition 3.2** (16). Let $M$ be a semi-invariant $\xi^\perp$-submanifold of an $LP$-Sasakian manifold $\bar{M}$. Then

(a) $BN \in D^\perp$, 


(b) $CN \in \mu$

for any $N \in \Gamma(TM^\perp)$.

**Proposition 3.3.** Let $M$ be a semi-invariant $\xi^\perp$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then

$$A_{\omega Z}W = A_{\omega W}Z.$$ 

**Proof** Let $Y, Z \in D^\perp$. Using (2.5), (3.2), (3.4) and (3.6), we have

$$g(A_{\phi W}Z, X) = g(h(X, Z), \phi W)$$

$$= g(\nabla_X Z, \phi W)$$

$$= g(\phi \nabla_X Z, W)$$

$$= g(\nabla_X \phi Z, W)$$

$$= -g(\phi Z, \nabla_X W)$$

$$= -g(h(X, W), \phi Z)$$

$$= -g(A_{\phi Z}W, X),$$

which is equivalent to

$$A_{\phi W}Z = A_{\phi Z}W.$$ 

But from (3.2), we have $\phi Z = \omega Z$ and $\phi W = \omega W$, then above equation reduces to $A_{\omega W}Z = A_{\omega Z}W$.

**Theorem 3.1.** Let $M$ be a semi-invariant $\xi^\perp$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then the distribution $D$ is integrable if and only if

$$h(X, \phi Y) = h(Y, \phi X)$$  \hspace{1cm} (3.10)

$\forall X, Y \in \Gamma(D)$.

**Proof** Let $X, Y \in \Gamma(D)$. Then from (3.7)(b), we get

$$\omega[X, Y] = h(X, tY) - h(Y, tX).$$  \hspace{1cm} (3.11)

Our assertion is a consequence of (3.11).
**Theorem 3.2.** Let $M$ be a semi-invariant $\xi^\perp$-submanifold of an LP-Sasakian manifold $\tilde{M}$. Then the distribution $D^\perp$ is integrable.

**Proof** In view of (3.7)(a) and Proposition 3.3, letting $Z, W \in \Gamma(D^\perp)$, we have


Consequently, $[Z, W] \in \Gamma(D^\perp)$ for all $Z, W \in \Gamma(D^\perp)$. Hence $D^\perp$ is integrable.

Suppose that $(e_i, \phi e_i, e_{2p+j}), i \in 1, 2, ..., p, j \in 1, 2, ..., q$ be an adapted orthonormal local frame on $M$, where $q = \dim D^\perp$. Now, we can state the following:

**Theorem 3.3.** Let $M$ be a semi-invariant $\xi^\perp$-submanifold of an LP-Sasakian manifold $\tilde{M}$. Then

$$\eta(H) = 1/m \text{ trace}(A_\xi), \ m = 2p + q.$$ 

**Proof** From the general mean curvature formula $H = 1/m \sum_{a=1}^s \text{ trace}(A_{\xi_a})\xi_a$, where $\{\xi_1, \xi_2, ..., \xi_s\}$ is an orthonormal basis in $TM^\perp$, the conclusion holds by straightforward computations.

**Theorem 3.4.** Let $M$ be a semi-invariant $\xi^\perp$-submanifold of an LP-Sasakian manifold $\tilde{M}$. Then

(1) if the distribution $D$ is integrable, then its leaves are totally geodesic in $M$ if and only if $h(X, Y) \in \Gamma(\mu), \ X, Y \in \Gamma(D)$,

(2) any leaf of the distribution $D^\perp$ is totally geodesic in $M$ if and only if $h(X, Z) \in \Gamma(\mu), \ X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$.

**Proof** Let us prove the first statement. Let $M^*$ be a leaf of the integrable distribution $D$ and $h^*$ the second fundamental form of $M^*$ in $M$. Also, let $X, Y \in M^*, \ X, Y \in D$.

Differentiating covariantly $\phi Y = tY$ and using (3.4), we get

$$\nabla_X tY + h^*(X, tY) = (\nabla_X \phi)Y + \phi(\tilde{\nabla}_X Y).$$ 

Using (2.5) in above equation, we have

$$(\nabla_X tY) + h^*(X, tY) = g(X, Y) \xi + \eta(Y)X + 2\eta(X) \eta(Y)\xi + \phi(\tilde{\nabla}_X Y).$$
Taking inner product with $Z$ and noting that $Z \in D^\perp$, $\phi Z \in \phi D^\perp \subset TM^\perp$, $g(\phi X, Y) = g(X, \phi Y)$, we get

$$g(h^*(X, tY), Z) = g(\phi \nabla_X Y, Z)$$
$$g(h^*(X, tY), Z) = g(\nabla_X Y, \phi Z)$$
$$g(h^*(X, tY), Z) = g(\nabla_X Y + h(X, Y), \phi Z)$$
$$g(h^*(X, tY), Z) = g(\nabla_X Y, \phi Z) + g((h(X, Y), \phi Z)$$
$$g(h^*(X, tY), Z) = g(h(X, Y), \phi Z),$$

which gives

$$h^*(X, tY) = 0,$$

if and only if $h(X, Y) \in \mu$.

The proof of second part of the theorem is analogous to that of Kenmotsu case in ([15], P. 117).

4. Totally umbilical semi-invariant $\xi^\perp$-submanifolds

In this section, we obtain a complete characterization of a totally umbilical semi-invariant $\xi^\perp$-submanifold of an LP-Sasakian manifold $\bar{M}$. For a totally umbilical submanifold we have

$$h(X, Y) = g(X, Y)H, \ X, Y \in \Gamma(TM). \tag{4.1}$$

**Theorem 4.1.** A semi-invariant $\xi^\perp$-submanifold $M$ of an LP-Sasakian manifold $\bar{M}$ with $\dim D^\perp \geq 2$ is totally umbilical if and only if

$$h(X, Y) = 1/m \ g(X, Y) \ \text{trace} \ (A_\xi)\xi. \tag{4.2}$$

**Proof** Suppose that $M$ is a totally umbilical semi-invariant $\xi^\perp$-submanifold of an LP-Sasakian manifold $\bar{M}$. Let $X \in \Gamma(D)$ be the unit vector field and $N \in \Gamma(\mu)$. Using Gauss formula (3.4), we get

$$h(X, X) = -\nabla_X X + \phi(\bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi)X) - \eta(\nabla_X X)\xi.$$  

$$= -\nabla_X X + \phi(\bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi)X) - g(\nabla_X X + h(X, X), \xi)\xi.$$  

$$= -\nabla_X X + \phi(\bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi)X)$$

Taking inner product with $N$, we have

$$g(H, N) = g(h(X, X), N) = 0,$$
which shows that $H \in \phi D^\perp \oplus \text{span} \{\xi\}$.

Now, letting $W,Z \in D^\perp$, From (2.5) and (3.5), we get

$$g(W,Z)\xi + \phi(\nabla_W Z + \phi h(W,Z)) = -A_\phi ZW + \nabla^\perp_W \phi Z.$$ 

Equating vertical components of above equations and then the inner product with $\phi H$ gives

$$g(W,Z)g(\phi H,\phi H) = g(Z,\phi H)g(W,\phi H). \quad (4.3)$$

Since $D^\perp \geq 2$, for $Z = W \perp \phi H$, the above relation gives $\phi H = 0$ which implies that $H \in \text{span}\{\xi\}$. If we consider an orthonormal frame $\{e_i,e_{p+i}\}, i = 1,2,3,\ldots,p$ on $M$. Since $M$ is a semi-invariant $\xi^\perp$-submanifold, we can write

$$H = g(H,\xi)\xi = 1/m \sum g(h(e_i,e_i),\xi)\xi = 1/m \text{trace}(A_\xi)\xi.$$ 

Using (4.1) in above equation, we get (4.2).

Conversely, if (4.2) holds, then we get (4.3). From (4.2) and (4.3) together we conclude that $M$ is totally umbilical.

**Corollary 4.1.** Every semi-invariant $\xi^\perp$-hypersurface $M$ of an LP-Sasakian manifold is geodesic.

*Proof* Let $M$ is a hypersurface, that is $TM^\perp = \text{span}\{\xi\}$, which implies that $h(X,Y) \in \text{span} \xi$. Then Corollary 4.2 follows from (4.3).

We call a semi-invariant product as a semi-invariant $\xi^\perp$-submanifold of $\bar{M}$ which can be locally written as a Riemannian product of a $\phi$-invariant submanifold and a $\phi$ anti–invariant submanifold of $\bar{M}$, both of them orthogonal to $\xi$.

**Theorem 4.2.** Let $M$ be a totally umbilical semi-invariant $\xi^\perp$-submanifold of an LP-Sasakian manifold $\bar{M}$ with $\dim D^\perp \geq 2$. Then $M$ is a semi-invariant product.

*Proof* Let $M$ be a totally umbilical submanifold, then $h(X,Z) = 0$ for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. So by Theorem 3.4, the leaves of $D^\perp$ are totally geodesic submanifold of $M$. By
Corollary 4.1, \( h(X,Y) \in \text{span} \{\xi\} \subseteq \mu \) for any \( X,Y \in \Gamma(D) \). Combining this fact with Theorem 3.4, this implies that the invariant distribution \( D \) is integrable and its integral manifolds are totally geodesic submanifolds of \( M \). Hence we conclude that \( M \) is semi-invariant product.

**Theorem 4.3.** Let \( M \) be a totally umbilical semi-invariant \( \xi^\perp \)-submanifold of an LP-Sasakian manifold \( \bar{M} \). If \( D \) is integrable, then each leaf of \( D \) is a totally geodesic submanifold of \( M \).

**Proof** Using (3.7)(b) for any \( X \in \Gamma(D) \), we get

\[
\omega(\nabla_X X) = -g(X,X)CH + g(X,\phi X)H - g(X,X)\xi.
\]

Since \( CH \in \Gamma(\mu) \) by Proposition 3.2, \( H \in \text{span} \{\xi\} \) from Theorem 4.1, \( \xi \in \Gamma(\mu) \) and \( \omega(\nabla_X X) \in \phi D^\perp \). From the above equation we deduce that \( \omega(\nabla_X X) = 0 \), or equivalently

\[
\nabla_X X \in \Gamma(D) \quad \forall \quad X \in \Gamma(D).
\]  

(4.4)

As \( D \) is integrable, Frobenius theorem ensures that \( M \) is foliated by leaves of \( D \). Combining this fact with (4.4), we conclude that the leaves of \( D \) are totally geodesic submanifolds of \( M \).

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**References**


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