ALMOST G-CONTACT METRIC MANIFOLD

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Abstract. In this paper, starting from only a global basis of vector fields, we construct a class of almost contact metric manifolds and we give concrete example. Next, we study some essential types belonging to this class on dimension 3 and we construct several examples.

1. Introduction

Fortunately, the rich theory of vector spaces endowed with a Euclidean inner product can, to a great extent, be lifted to various bundles associated with a manifold. The notion of global (and local) frame plays an important technical role. It should be mentioned however that a global basis of $\mathfrak{X}(M)$ (the Lie algebra of smooth vector fields on a manifold $M$) i.e., $n$ vector fields that are linearly independent over $\mathcal{F}(M)$ and span $\mathfrak{X}(M)$, does not exist in general. Manifolds that do admit such a global basis for $\mathfrak{X}(M)$ are called parallelizable. It is straightforward to show that a finite-dimensional manifold is parallelizable if and only if its tangent bundle is trivial (that is, isomorphic to the product, $M \times \mathbb{R}^n$).

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As an illustration, we can prove that the tangent bundle, $TS^1$, of the circle, is trivial. Indeed, we can find a section that is everywhere nonzero, i.e. a non-vanishing vector field, namely

$$X(\cos \theta, \sin \theta) = (-\sin \theta, \cos \theta).$$

The reader should try proving that $TS^3$ is also trivial (use the quaternions). However, $TS^2$ is nontrivial, although this not so easy to prove.

More generally, it can be shown that $TS^n$ is nontrivial for all even $n \geq 2$. It can even be shown that $S^1, S^3$ and $S^7$ are the only spheres whose tangent bundle is trivial. This is a rather deep theorem and its proof is hard.

Here, starting from a Global frame we construct a class of almost contact metric structures, specifically, many well-known almost contact metric structures (Sasakian, cosymplectic, Kenmotsu) in dimension three and we confirm the construction each time with a concrete example showing that the case is non-vacuous.

This work is organized in the following way:

Section 2 is devoted to the background of the structures which will be used in the sequel. In Section 3 we give the necessary techniques to construct an almost contact metric structure from a global frame of vector fields and we give an example. In Section 4 we focus on the case of three-dimensional and we show how to construct some basic structures with concrete examples.

### 2. Review of needed notions

An odd-dimensional Riemannian manifold $(M^{2n+1}, g)$ is said to be an almost contact metric manifold if there exist on $M$ a $(1, 1)$-tensor field $\varphi$, a vector field $\xi$ (called the structure vector field) and a 1-form $\eta$ such that

$$\begin{cases} 
(1) : \eta(\xi) = 1, \\
(2) : \varphi^2(X) = -X + \eta(X)\xi, \\
(3) : g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). 
\end{cases} \tag{2.1}$$

for any vector fields $X, Y$ on $M$. In particular, in an almost contact metric manifold we also have

$$\varphi \xi = 0 \quad \text{and} \quad \eta \circ \varphi = 0. \tag{2.2}$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Omega$, where $\Omega(X, Y) = g(X, \varphi Y)$ is called the fundamental 2-form of $M$. If, in addition, $\xi$ is a Killing vector field,
then $M$ is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field $X$ on $M$.

On the other hand, the almost contact metric structure of $M$ is said to be normal if

$$N_\varphi(X,Y) = [\varphi,\varphi](X,Y) + 2d\eta (X,Y)\xi = 0,$$

(2.3)

for any $X, Y$, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of $\varphi$, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y]$$

An almost contact metric structures $(\varphi,\xi,\eta,g)$ on $M$ is said to be:

\begin{align*}
(a) : \text{Sasaki} & \iff \Omega = d\eta \text{ and } (\varphi,\xi,\eta) \text{ is normal}, \\
(b) : \text{Cosymplectic} & \iff d\Omega = d\eta = 0 \text{ and } (\varphi,\xi,\eta) \text{ is normal}, \\
(c) : \text{Kenmotsu} & \iff d\eta = 0, \ d\Omega = 2\eta \wedge \Omega \text{ and } (\varphi,\xi,\eta) \text{ is normal}.
\end{align*}

(2.4)

where $d$ denotes the exterior derivative.

These manifolds can be characterized through their Levi-Civita connection, by requiring

\begin{align*}
(1) : \text{Sasaki} & \iff (\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \\
(2) : \text{Cosymplectic} & \iff \nabla \varphi = 0, \\
(3) : \text{Kenmotsu} & \iff (\nabla_X \varphi)Y = g(\varphi X,Y)\xi - \eta(Y)\varphi X.
\end{align*}

(2.5)

For more background on almost contact metric manifolds, we recommend the reference [1], [2], [3] and [5].

### 3. Almost G-Contact Metric Manifold

Let $\{e_0, e_i\}_{1\leq i\leq 2n}$ be the global frame of vector fields and $\{\theta^0, \theta^i\}_{1\leq i\leq 2n}$ be the dual frame of differential 1-forms on a $(2n+1)$-dimensional smooth manifold $M$. Define a $(1,1)$-tensor field $\varphi$ on $M$ by

$$\varphi = \sum_{i=1}^{n} e_{2i} \wedge e_{2i-1} = \sum_{i=1}^{n} (\theta^{2i-1} \otimes e_{2i} - \theta^{2i} \otimes e_{2i-1}),$$

(3.6)

i.e. for all vector field $X$ on $M$, we have

$$\varphi X = \sum_{i=1}^{n} (e_{2i} \wedge e_{2i-1})X$$

$$= \sum_{i=1}^{n} (g(e_{2i-1},X)e_{2i} - g(e_{2i},X)e_{2i-1})$$

$$= \sum_{i=1}^{n} \theta^{2i-1}(X)e_{2i} - \theta^{2i}(X)e_{2i-1},$$
and a Riemannian metric $g$ on $M$ with $\{e_i\}_{0 \leq i \leq 2n+1}$ is an orthonormal frame, so that

$$g = \sum_{i=0}^{2n} \theta^i \otimes \theta^i. \quad (3.7)$$

With these identities, we state the following:

**Theorem 3.1.** The manifold $(M, \varphi, e_0, \theta^0, g)$ defined as above is an almost contact metric manifold.

**Proof.** According to the conditions $(2.1)$, the data $(M, g, \varphi, e_0, \theta^0, g)$ is an almost contact metric manifold if only two conditions are satisfied

$$\varphi^2 X = -X + \theta^0(X)e_0 \quad \text{and} \quad g(\varphi X, \varphi Y) = g(X, Y) - \theta^0(X)\theta^0(Y).$$

Using formula $(3.6)$ we get

$$\varphi e_{2i} = -e_{2i-1} \quad \text{and} \quad \varphi e_{2i-1} = e_{2i}.$$ 

To prove the first condition, we have for all $X$ vectors field on $M$

$$\varphi^2 X = \sum_{i=1}^{n} \left( \theta_{2i-1}(X)\varphi e_{2i} - \theta^{2i}(X)\varphi e_{2i-1} \right)$$

$$= -\sum_{i=1}^{n} \left( \theta^{2i-1}(X)e_{2i-1} + \theta^{2i}(X)e_{2i} \right)$$

$$= -\sum_{i=1}^{2n} \theta^i(X)e_i$$

$$= -X + \theta^0(X)e_0.$$ 

For the second condition, for all $X$ and $Y$ vectors fields on $M$ we have

$$g(\varphi X, \varphi Y) = \sum_{i=1}^{n} \left( \theta^{2i-1}(X)\theta^{2i-1}(Y) + \theta^{2i}(X)\theta^{2i}(Y) \right)$$

$$= \sum_{i=1}^{2n} \theta^i(X)\theta^i(Y)$$

$$= g\left(X, \sum_{i=1}^{2n} \theta^i(Y)e_i \right)$$

$$= g(X, Y - \theta^0(Y)e_0)$$

$$= g(X, Y) - \theta^0(X)\theta^0(Y),$$

which completes the proof.

We refer to this construction as **almost G-contact metric manifold**.
Example 3.1. Let \((x_i)\) be the Cartesian coordinates in \(\mathbb{R}^5\) and \(\partial_i = \frac{\partial}{\partial x_i}\). Define a global frame of vector fields on \(\mathbb{R}^5\) by:

\[
e_0 = \partial_5, \quad e_1 = \partial_1 + f \partial_5, \quad e_2 = \partial_2, \quad e_3 = \partial_3 + h \partial_5, \quad e_4 = \partial_4,
\]

where \(f, h\) are two strictly positive functions on \(\mathbb{R}^5\) and let \(g\) be the Riemannian metric defined by

\[
g(e_i, e_j) = \delta_{ij} \quad \forall i, j \in \{0, \ldots, 5\},
\]

that is, the form of the metric becomes

\[
g = \begin{pmatrix}
1 + f^2 & 0 & fh & 0 & -f \\
0 & 1 & 0 & 0 & 0 \\
fh & 0 & 1 + h^2 & 0 & -h \\
0 & 0 & 0 & 1 & 0 \\
-f & 0 & -h & 0 & 1
\end{pmatrix},
\]

and the 1-form corresponding to \(e_0\) is \(\theta^0 = -f \, dx_1 - h \, dx_3 + dx_5\).

To define \(\varphi\), let us use the formula

\[
\varphi = \sum_{i=1}^{2} e_{2i} \wedge e_{2i-1}
\]

\[
= e_2 \wedge e_1 + e_4 \wedge e_3
\]

\[
= \theta^1 \otimes e_2 - \theta^2 \otimes e_1 + \theta^3 \otimes e_4 - \theta^4 \otimes e_3,
\]

we get

\[
\varphi = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -f & 0 & -h & 0
\end{pmatrix},
\]

where we can check that \((\mathbb{R}^5, \varphi, e_0, \theta^0, g)\) is an almost G-contact metric manifold.

Remark 3.1. Any almost G-contact metric manifold is an almost contact metric manifold, the converse is not true in general.

While this is an area of possible future research we mention briefly that one easily has the following:
The fundamental 2-form $\Phi$ of $(\varphi, e_0, \theta^0, g)$ is:

$$
\Phi(X,Y) = g(X, \varphi Y)
$$

$$
= g \left( X, \sum_{i=1}^{n} (e_{2i} \wedge e_{2i-1}) Y \right)
$$

$$
= g \left( X, \sum_{i=1}^{n} \theta^{2i-1}(Y) e_{2i} - \theta^{2i}(Y) e_{2i-1} \right)
$$

$$
= \sum_{i=1}^{n} \left( \theta^{2i-1}(Y) \theta^{2i}(X) - \theta^{2i}(Y) \theta^{2i-1}(X) \right)
$$

$$
= 2 \sum_{i=1}^{n} (\theta^{2i} \wedge \theta^{2i-1})(X,Y),
$$

we can check that is very simply as follows:

$$
\Phi = 2 \theta^{2i} \wedge \theta^{2i-1}. \quad (3.8)
$$

**Proposition 3.1.** Let $(M, \Phi, e_0, \theta^0, g)$ be an almost $G$-contact manifold. Then, we have

$$
d\Phi = \frac{\alpha}{n} \theta^0 \wedge \Phi,
$$

where $d$ denote the exterior derivative and

$$
\alpha = \text{div} e_0 + \sum_{i<j} \left( (\mathcal{L}_{e_0} g)(e_{2i-1}, e_{2j-1}) + (\mathcal{L}_{e_0} g)(e_{2i}, e_{2j}) \right).
$$

**Proof.** Let $U = \sum_{i=1}^{n} e_{2i-1}$ and $V = \sum_{i=1}^{n} e_{2i}$ two vectors fields on $M$. Putting $d\Phi = \sigma \theta^0 \wedge \Phi$ for a certain functions $\sigma$ on $M$. Then, we get

$$
3 (\theta^0 \wedge \Phi)(e_0, U, V) = n,
$$

$$
3d\Phi(e_0, U, V) = (\nabla_{e_0} \Phi)(U, V) + (\nabla_U \Phi)(V, \xi) + (\nabla_V \Phi)(\xi, U)
$$

$$
= -\Phi(V, \nabla_U \theta^0) - \Phi(\nabla_V \theta^0, U)
$$

$$
= \sum_{i=1}^{n} (\theta^{2i-1}(\nabla_U \theta^0) + \theta^{2i}(\nabla_V \theta^0))
$$

$$
= g(\nabla_U \theta^0, U) + g(\nabla_V \theta^0, V)
$$

$$
= \text{div} \xi + \sum_{i<j} \left( (\mathcal{L}_{e_0} g)(e_{2i-1}, e_{2j-1}) + (\mathcal{L}_{e_0} g)(e_{2i}, e_{2j}) \right)
$$

$$
= \alpha,
$$

which implies $\sigma = \frac{\alpha}{n}$. 
4. 3-DIMENSIONAL ALMOST G-CONTACT METRIC MANIFOLD

Let \( \{e_0, e_1, e_2\} \) be the global frame of vector fields and \( \{\theta^0, \theta^1, \theta^2\} \) be the dual frame of differential 1-forms on a 3-dimensional smooth manifold \( M^3 \). Define a \((1,1)\)-tensor field \( \varphi \) on \( M \) by

\[
\varphi = e_2 \wedge e_1 = \theta^1 \otimes e_2 - \theta^2 \otimes e_1
\]

and a Riemannian metric \( g \) on \( M \) with \( \{e_i\}_{0 \leq i \leq 2} \) is an orthonormal frame, so that

\[
g = \sum_{i=0}^{2} \theta^i \otimes \theta^i.
\]

According to the theorem \[3.1\] \((M^3, \varphi, e_0, \theta^0, g)\) is an almost G-contact metric manifold.

Through the rest of this paper, we are mainly interested in dimension three. Below we recall certain results concerning this case.

For an arbitrary 3-dimensional almost contact metric manifold \((M^3, \xi, \eta, g)\), we have

\[
d\Phi = 2\alpha\eta \wedge \Phi.
\]

A 3-dimensional almost contact metric manifold \( M \) is normal if and only if for all \( X \) vectors field on \( M \),

\[
\nabla_{\varphi X} \xi = \varphi \nabla_X \xi,
\]

or, equivalently,

\[
\nabla_X \xi = -\alpha \varphi^2 X - \beta \varphi X,
\]

and for a normal almost contact metric manifold \( M \) we have \([4], Corollary 1\)

\[
\nabla_\xi \xi = 0 \quad \text{and} \quad d\eta = \beta \Phi.
\]

where \( \alpha \) and \( \beta \) are the functions defined by \( 2\alpha = \text{div} \xi \) and \( 2\beta = \text{tr}(\varphi \nabla \xi) \) and \( \nabla \) is the Levi-Civita connection on \( M \).

From formulas (2.4) and (4.12)-(4.15), one can easily proof that

\[
\begin{align*}
(a) & : \text{Sasaki} \iff \alpha = 0, \ \beta = 1 \ \text{and} \ (\varphi, \xi, \eta) \ \text{is normal}, \\
(b) & : \text{Cosymplectic} \iff \alpha = \beta = 0 \ \text{and} \ (\varphi, \xi, \eta) \ \text{is normal}, \\
(c) & : \text{Kenmotsu} \iff \alpha = 1, \ \beta = 0 \ \text{and} \ (\varphi, \xi, \eta) \ \text{is normal}.
\end{align*}
\]

As a consequence of the above formulas (2.5), we immediately obtain the following result:
**Theorem 4.1.** A 3-dimensional almost $G$-contact metric manifold is:

\[
\begin{cases}
(1): G - \text{Sasaki} \iff \nabla_X e_0 = -\varphi X, \\
(2): G - \text{cosymplectic} \iff \nabla_X e_0 = 0, \\
(3): G - \text{Kenmotsu} \iff \nabla_X e_0 = -\varphi^2 X.
\end{cases}
\tag{4.17}
\]

for all vectors fields $X$ on $M$.

**Proof.** According to the cases given in formulas (2.5) we have:

(1): An almost $G$-contact metric manifold is Sasakian if and only if

\[
(\nabla_X \varphi)Y = g(X,Y)e_0 - \theta^0(Y)X,
\tag{4.18}
\]

taking $Y = e_0$ with $\theta^0(\nabla_X e_0) = 0$, we obtain

\[
(\nabla_X \varphi)e_0 = \theta^0(X)e_0 - X \iff -\varphi \nabla_X e_0 = \theta^0(X)e_0 - X
\]

\[
\iff -\varphi^2 \nabla_X e_0 = -\varphi X
\]

\[
\iff \nabla_X e_0 = -\varphi X,
\]

we proved that if $M$ is $G$-Sasakian then $\nabla_X e_0 = -\varphi X$. Conversely, suppose that

\[
\nabla_X e_0 = -\varphi X.
\tag{4.19}
\]

It is easy to see that $\nabla_{\varphi X} e_0 = \varphi \nabla_X e_0$, then the structure $(\varphi, e_0, \theta^0)$ is normal and also we have (see [4], Prop. 2)

\[
\nabla_X e_0 = -\alpha \varphi^2 X - \beta \varphi X.
\tag{4.20}
\]

From formulas (4.19) and (4.20), we get

\[
\alpha = 0 \quad \text{and} \quad \beta = 1,
\]

following formulas (4.16), $M$ is a $G$-Sasakian manifold.

(2): An almost $G$-contact metric manifold is cosymplectic if and only if

\[
(\nabla_X \varphi)Y = 0,
\tag{4.21}
\]

taking $Y = e_0$, we obtain $\nabla_X e_0 = 0$.

Conversely, suppose that

\[
\nabla_X e_0 = 0.
\tag{4.22}
\]
It is easy to see that \( \nabla_{\varphi X}e_0 = \varphi \nabla_X e_0 = 0 \), then the structure \((\varphi, e_0, \theta^0)\) is normal, using formulas (4.14), we get
\[
\alpha = \beta = 0,
\]
following formulas (4.16), \(M\) is a G-cosymplectic manifold.

(3): An almost G-contact metric manifold is Kenmotsu if and only if
\[
(\nabla_X \varphi)Y = g(\varphi X, Y)e_0 - \theta^0(Y)\varphi X,
\]
(4.23)
taking \(Y = e_0\), we get \(\nabla_X e_0 = -\varphi^2 X\). we proved that if \(M\) is G-Kenmotsu then \(\nabla_X e_0 = -\varphi^2 X\). Conversely, suppose that
\[
\nabla_X e_0 = -\varphi^2 X.
\]
(4.24)
we obtain
\[
\nabla_{\varphi X} e_0 = -\varphi^3 X = \varphi \nabla_X e_0,
\]
therefore, the structure \((\varphi, e_0, \theta^0)\) is normal and also we have (see [4], Prop. 2)
\[
\nabla_X e_0 = -\alpha \varphi^2 X - \beta \varphi X.
\]
(4.25)
From formulas (4.24) and (4.25), we get
\[
\alpha = 1 \quad \text{and} \quad \beta = 0,
\]
following formulas (4.16), \(M\) is a G-Kenmotsu manifold.

5. Examples

Let \((x, y, z)\) denote the Cartesian coordinates in \(\mathbb{R}^3\). We denote the global frame of vector fields on \(\mathbb{R}^3\) by \((e_0, e_1, e_2)\) and the dual frame of differential 1-forms by \((\theta^0, \theta^1, \theta^2)\) such that \(\theta^i(e_j) = \delta_{ij}\) for all \(i, j \in \{0, 1, 2\}\).

Example 5.1. \((G\text{-Sasakian manifold})\)

Consider
\[
\theta^0 = dx + 2zdy, \quad \theta^1 = dy, \quad \theta^2 = dz,
\]
and
\[
e_0 = \frac{\partial}{\partial x}, \quad e_1 = -2z \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial z}.
\]
For the non-zero Lie brackets of \((e_i)\), we have:
\[
[e_1, e_2] = 2e_0.
\]
Define an almost contact structure \((\varphi, e_0, \theta^0)\) on \(M\) by assuming 

\[
\varphi e_0 = 0, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1.
\]

Let \(g\) be the Riemannian metric on \(M\) for which \((e_i)\) is an orthonormal frame, so that 

\[
g = \sum \theta^i \otimes \theta^i.
\]

It is obvious that \((\varphi, e_0, \theta^0, g)\) is an almost contact metric structure on \(\mathbb{R}^3\).

For the Levi-Civita connection corresponding to \(g\), we have 

\[
\nabla_{e_0} e_0 = \nabla_{e_1} e_1 = \nabla_{e_2} e_2 = 0, \quad \nabla_{e_0} e_1 = \nabla_{e_1} e_0 = -e_2,
\]

\[
\nabla_{e_0} e_2 = \nabla_{e_2} e_0 = e_1, \quad \nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = e_0.
\]

We can easily check that \(i \in \{0, 1, 2\}\)

\[
\nabla_{e_i} e_0 = -\varphi e_i.
\]

Knowing that \((\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y\) for all \(X\) and \(Y\) vectors fields on \(M\), one can check that 

\[
(\nabla_{e_i} \varphi) e_j = \delta_{ij} e_0 - \theta^0(e_j) e_i
\]

for all \(i, j \in \{0, 1, 2\}\). Therefore, \((\mathbb{R}^3, \varphi, e_0, \theta^0, g)\) is a G-Sasakian manifold.

**Example 5.2. (G-cosymplectic manifold)**

For the global frame

\[
e_0 = \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}, \quad e_1 = \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial z},
\]

we define a Riemannian metric \(g\) by

\[
g(e_0, e_1) = g(e_0, e_2) = g(e_1, e_2) = 0,
\]

\[
g(e_0, e_0) = g(e_1, e_1) = g(e_2, e_2) = 1
\]

that is, the form of the metric becomes

\[
g = \begin{pmatrix}
1 + x^2 & 0 & -x \\
0 & 1 & 0 \\
-x & 0 & 1
\end{pmatrix},
\]

and the corresponding 1-forms are 

\[
\theta^0 = dx, \quad \theta^1 = dy, \quad \theta^2 = -xdx + dz,
\]
To define $\varphi$, let’s use the formula $\varphi = e_2 \wedge e_1$, we get

$$
\varphi = \begin{pmatrix}
0 & 0 & 0 \\
x & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}.
$$

where we can check that $(\varphi, \xi, \eta, g)$ is an almost G-contact metric structure on $E^3$.

It is easy to see that for all $i, j \in \{0, 1, 2\}$,

$$[e_i, e_j] = 0,$

therefore, all components of the Levi-Civita connection are zero. Then, for all $i \in \{0, 1, 2\}$ we obtain

$$\nabla_{e_i} e_0 = 0,$

which shows that $(\mathbb{R}^3, \varphi, e_0, \theta^0, g)$ is a G-cosymplectic manifold. One can verify this result by classical reasoning, using formulas (4.16).

**Example 5.3. (G-Kenmotsu manifold)**

Consider

$$
\theta^0 = -xdx + dz, \quad \theta^1 = e^zdx, \quad \theta^2 = e^zdy,
$$

and

$$e_0 = \frac{\partial}{\partial z}, \quad e_1 = e^{-z} \left( \frac{\partial}{\partial x} + x \frac{\partial}{\partial z} \right), \quad e_2 = e^{-z} \frac{\partial}{\partial y}.
$$

For the non-zero Lie brackets of $(e_i)$, we have:

$$[e_0, e_1] = -e_1, \quad [e_0, e_2] = -e_2, \quad [e_1, e_2] = -xe^{-z}e_2.
$$

Define an almost contact structure $(\varphi, e_0, \theta^0)$ on $M$ by assuming

$$
\varphi e_0 = 0, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1.
$$

Let $g$ be the Riemannian metric on $M$ for which $(e_i)$ is an orthonormal frame, so that $g = \sum \theta^i \otimes \theta^i$. It is obvious that $(\varphi, e_0, \theta^0, g)$ is an almost contact metric structure on $\mathbb{R}^3$.

For the Levi-Civita connection corresponding to $g$, we have

$$
\nabla_{e_0} e_0 = \nabla_{e_0} e_1 = \nabla_{e_0} e_2 = \nabla_{e_1} e_2 = 0,
$$

$$
\nabla_{e_1} e_0 = e_1, \quad \nabla_{e_2} e_0 = e_2, \quad \nabla_{e_1} e_1 = e_0
$$

$$
\nabla_{e_2} e_1 = xe^{-z}e_2, \quad \nabla_{e_2} e_2 = -e_0 - xe^{-z}e_1.
$$
We can see that for all $i \in \{0, 1, 2\}$

$$\nabla_{e_i} e_0 = -\varphi^2 e_i.$$

Knowing that $(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y$, one can check that

$$(\nabla_X \varphi)Y = g(\varphi X, Y)e_0 - \theta^0(Y)\varphi X,$$

for all $X, Y \in \{e_0, e_1, e_2\}$. Therefore, $(\mathbb{R}^3, \varphi, e_0, \theta^0, g)$ is a G-Kenmotsu manifold.

**References**


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