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ALMOST POLY-NORDEN MANIFOLDS

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Abstract. In this paper, poly-Norden manifolds are introduced as a new type of manifolds. We show that this class includes Norden manifolds and Euclidean $n-$ space as examples. We investigate certain geometric properties of poly-Norden manifolds and obtain conditions for a holomorphic-like map between such manifolds to be totally geodesic. We also investigate constancy of certain maps between poly-Norden manifolds and other manifolds endowed with differential structures.

1. Introduction

Manifolds equipped with certain differential-geometric structures have been studied widely in differential geometry. Indeed, almost complex manifolds, almost contact manifolds and almost product manifolds have been studied extensively by many authors. Recently, by inspiring the Golden ratio, Golden Riemannian manifolds were introduced in [5]. In [11], the authors have also introduced metallical Riemannian manifolds by inspiring metallic mean which is a generalization of Golden mean, Silver mean etc. Both manifolds have been studied by many authors [6], [7],[9],[10], [11], [12],[15], [14], [16] and [19]. We note that metallic mean was first defined by de Spinadel [17].

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On the other hand, in [13], the author has defined Bronze mean which is different from the Bronze mean given in [11]. We also note that there is no inclusion relation between the Bronze mean defined in [13] and metallic mean. In [13], the author introduced the Bronze Fibonacci and Lucas numbers. He also investigated the relationship between the convergents of the continued fractions of the powers of the Bronze means and the Bronze Fibonacci and Lucas numbers.

In this paper, by inspiring from [5] and [13], we introduce almost poly-Norden manifolds and investigate the geometry of such manifolds. First of all, we observe that such manifolds include some well known manifolds; Norden manifolds and Euclidean spaces. We then investigate certain properties of this structure and obtain constancy of certain maps.

2. Preliminaries

In this section, we gather main tools needed for the paper.

2.1. **A new Bronze mean.** We will not repeat Golden ratio, Fibonacci number or Lucas numbers which are well known. But we will give brief information on a new mean and related numbers introduced in [13]. The Bronze mean is defined by

$$B_m = \frac{m + \sqrt{m^2 - 4}}{2} \tag{2.1}$$

which is the positive solution of the equation

$$x^2 - mx + 1 = 0.$$

The Bronze Fibonacci numbers $(f_{m,n})$ are a family of sequences defined by the recurrence $f_{m,n+2} = mf_{m,n+1} - f_{m,n}$, where $f_{m,0} = 0$ and $f_{m,1} = 1$. The Bronze Lucas numbers $(l_{m,n})$ are a family of sequence defined by the recurrence $l_{m,n+2} = ml_{m,n+1} - l_{m,n}$, where $l_{m,0} = 2$ and $l_{m,1} = m$. The continued fractions for the Bronze means are $\{m-1; \overline{1, m-2}\}$. The recurrence relation $B_m^{n+2} = mB_m^{n+1} - B_m^n$ is satisfied. The relations between Bronze Fibonacci numbers and Bronze Lucas numbers are

$$B_m^n = \frac{l_{m,n} + f_{m,n}\sqrt{m^2 - 4}}{2}.$$

Also note that the convergents of B_m^a are $\frac{f_{m,a(n+1)}}{f_{m,an}}$. Proofs of all above statements and more results can be found in [13].

2.2. **Totally geodesic maps.** Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \to N$ is a smooth mapping between them. Then the differential $d\varphi$ of φ can be viewed a section of the bundle $Hom\left(TM, \varphi^{-1}TN\right) \to M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $\left(\varphi^{-1}TN\right)_p = T_{\varphi(p)}N$, $p \in M$. $Hom\left(TM, \varphi^{-1}TN\right)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$\nabla d\varphi\left(X, Y\right) = \nabla^{\varphi}_X d\varphi\left(Y\right) - d\varphi\left(\nabla^M_X Y\right) \tag{2.2}$$

for $X, Y \in \Gamma\left(TM\right)$, where $\Gamma\left(TM\right)$ denotes the Lie algebra of the vector fields on M . It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \to (N, g_N)$ is said to be totally geodesic if $\nabla d\varphi = 0$. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths. For more information, see [1].

3. Almost poly-Norden manifolds

By inspiring from the Bronze mean (2.1) we introduce a new structure on a differentiable manifold M, namely, poly-Norden structure.

Definition 3.1. *Let M be a differentiable manifold. A poly-Norden structure on M is an $(1,1)$ tensor field Φ which satisfies the equation*

$$\Phi^2 = m\Phi - I \tag{3.3}$$

where I is the identity operator on the Lie algebra $\chi(M)$ of the vector fields on M. In this case, (M, Φ) is called an almost poly-Norden manifold.

We give an example of almost poly-Norden manifolds.

Example 3.1. *Consider the $4-$tuples real space $\mathbb{R}^4$ and define a map by*

$$\begin{array}{cccc} \Phi : & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \\ & (x_1, x_2, y_1, y_2) & & (B_m x_1, B_m x_2, \bar{B}_m y_1, \bar{B}_m y_2), \end{array}$$

where $B_m = \frac{m+\sqrt{m^2-4}}{2}$ and $\bar{B}_m = m - B_m$. Then it is easy to see that Φ satisfies $\Phi^2 = m\Phi - I$. Thus $(\mathbb{R}^4, \Phi)$ is an example of almost poly-Norden manifold.

We say that a semi-Riemannian metric g is $\Phi-$ compatible if

$$g(\Phi X, \Phi Y) = mg(\Phi X, Y) - g(X, Y) \tag{3.4}$$

for every $X, Y \in \chi(M)$. From this it follows that Φ is a self-adjoint operator with respect to g, i.e.,

$$g(\Phi X, Y) = g(X, \Phi Y) \tag{3.5}$$

Definition 3.2. *A semi-Riemannian manifold (M, g) endowed with a poly-Norden structure Φ so that the semi-Riemannian metric g is Φ-compatible is named an almost poly-Norden semi-Riemannian manifold and (g, Φ) is called an almost poly-Norden Riemannian structure on M.*

We now give an example of almost poly-Norden semi-Riemannian manifolds.

Example 3.2. *Let M be an almost complex manifold with almost complex structure J. A metric g is a Norden metric if $g(JX, JY) = -g(X, Y)$. If (M^{2n}, J) is an almost complex manifold with Norden metric g, Then (M^{2n}, J, g) is called an almost Norden manifold. Thus every almost Norden manifold is an almost poly-Norden semi-Riemannian manifold with $m = 0$.*

From now on, we will assume that the number m is different from zero throughout the article. We now investigate the geometry of an almost poly-Norden manifold.

Proposition 3.1. *Eigenvalues of an almost poly-Norden structure Φ are $\frac{m+\sqrt{m^2-4}}{2}$ and $\frac{m-\sqrt{m^2-4}}{2}$.*

Proposition 3.2. *An almost poly-Norden structure Φ is an isomorphism on a tangent space of M*

Since Φ is isomorphism, it has an inverse. Let us denote the inverse of Φ by $\tilde{\Phi}$, then we have

$$\tilde{\Phi} = -\Phi + mI. \tag{3.6}$$

We also have the following result.

Proposition 3.3. *$\tilde{\Phi}$ is not an poly-Norden structure on M.*

The following result shows that an almost complex structure determines a poly-Norden structure and vice versa.

Proposition 3.4. *Every complex structure on a semi-Riemannian manifold induces two poly-Norden structures given by*

$$\Phi_1 = \frac{m}{2}I + \frac{\sqrt{4-m^2}}{2}J,\ \Phi_2 = \frac{m}{2}I - \frac{\sqrt{4-m^2}}{2}J,\ -2 < m < 2$$

Proposition 3.5. *Every poly-Norden structure on a semi-Riemannian manifold induces two almost complex structures given by*

$$J_1 = \frac{-m}{\sqrt{4-m^2}}I + \frac{2}{\sqrt{4-m^2}}\Phi,\ J_2 = \frac{m}{\sqrt{4-m^2}}I - \frac{2}{\sqrt{4-m^2}}\Phi,\ -2 < m < 2$$

Next result implies that there are two orthogonal complementary distributions on almost polygonal semi-Riemannian manifold (M, ϕ, g).

Proposition 3.6. *On an almost poly-Norden semi-Riemannian manifold (M, ϕ, g), there are two complementary distributions D_l and $D_{l^\perp}$ corresponding to the projection operators*

$$l = \frac{1}{\sqrt{m^2-4}}(B_m I - \Phi),\ l^\perp = \frac{1}{\sqrt{m^2-4}}(-\bar{B}_m I + \Phi) \tag{3.7}$$

Corollary 3.1. *The complementary distributions D_l and $D_{l^\perp}$ are orthogonal with respect to the $\Phi-$compatible metric g, i.e. $g(D_l, D_{l^\perp}) = 0$.*

Definition 3.3. *Let (M, Φ, g) be an almost poly-Norden semi-Riemannian manifold. If the almost poly-Norden structure is parallel with respect to the Levi-Civita connectin ∇. Then (M, Φ, g) is called a poly-Norden semi-Riemannian manifold.*

One can see that $\nabla\Phi =$ is equivalent to $N_\Phi = 0$, where N_Φ is the Nijenhuis tensor field with respect to Φ, see:[3], [4], [8].

4. Poly-Norden maps between poly-Norden manifolds

In this section we give a new notion, namely poly-Norden map, and investigate conditions for a poly-Norden map to be totally geodesic. From now on we frequently denote an almost poly-Norden manifold by (M, Φ, m). We first give the following definion which is a version of a holomorphic map. Let φ be a map from an almost poly-Norden manifold (M, Φ_M, m) to an almost poly-Norden manifold (N, Φ_N, m'). Then we say that φ is a poly-Norden map if it satisfies $d\varphi\Phi_M = \Phi_N d\varphi$, where $d\varphi$ denotes the derivative map of φ. We provide the following elementary example.

Example 4.1. *Let* $\varphi : \mathbb{R}^4 \to \mathbb{R}^2$ *be a map defined by* $\varphi(x_1, x_2, x_3, x_4) = \left(\frac{x_1+x_2}{4}, \frac{x_3+x_4}{4}\right)$. *Then, by direct calculations*

$$\ker d\varphi = span\left\{X_1 = \partial_{x_1} - \partial_{x_2}, X_2 = \partial_{x_3} - \partial_{x_4}\right\}$$

and

$$(\ker d\varphi)^{\perp} = span\left\{Z_1 = \partial_{x_1} + \partial_{x_2}, Z_2 = \partial_{x_3} + \partial_{x_4}\right\}.$$

Then considering poly-Norden strucutures on $\mathbb{R}^4$ *and* $\mathbb{R}^2$ *defined by*

$$\Phi(x_1, x_2, x_3, x_4) = \left(B_m x_1, B_m x_2, \bar{B}_m x_3, \bar{B}_m x_4\right)$$

and

$$\Phi'(a_1, a_2) = \left(B_m a_1, \bar{B}_m a_2\right)$$

where B_m *and* $\bar{B}_m$ *are eigenvalues of poly-Norden structures. It is easy to see that* $d\varphi\,(\Phi Z_1) = \Phi' d\varphi\,(Z_1)$ *and* $d\varphi\,(\Phi Z_2) = \Phi' d\varphi\,(Z_2)$. *Thus* φ *is a poly-Norden map.*

From now on, when we mention a poly-Norden semi-Riemannian manifold in this section, we will assume that its almost poly-Norden structure is integrable.

Lemma 4.1. *Let* φ *be a poly-Norden map from a poly-Norden semi-Riemannian manifold* (M, Φ, g_M, m) *to a poly-Norden semi-Riemannian manifold* (N, Φ', g', m') *such that* $d\varphi\,\Phi = \Phi' d\varphi$ *is satisfied. Then we have*

$$(\nabla d\varphi)(\Phi X, \Phi Y) = m'\Phi'(\nabla d\varphi)(X, Y) - (\nabla d\varphi)(X, Y) \tag{4.8}$$

for $X, Y \in \Gamma(TM)$.

Proof. For $X, Y \in \Gamma(TM)$, from (2.13) and (2.11) we have

$$(\nabla d\varphi)(X, \Phi Y) = \nabla_X^{\varphi} d\varphi\,(\Phi Y) - d\varphi\left(\nabla_X^M \Phi Y\right).$$

Since $d\varphi\Phi = \Phi' d\varphi$ and both Φ and Φ' are integrable we have

$$(\nabla d\varphi)(X, \Phi Y) = \Phi'(\nabla d\varphi)(X, Y).$$

Using this equation and (3.3) we have the assertion.

We now give a necessary and sufficient condition for φ to be totally geodesic. We recall that a map φ is totally geodesic if $\nabla d\varphi = 0$. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths. From Lemma 4.1, we have the following result.

Theorem 4.1. *Let φ be an poly-Norden map from a poly-Norden semi-Riemannian manifold (M, Φ, g) to a poly-Norden semi-Riemannian manifold (N, Φ', g'). If one of the following conditions is satisfied, then φ is totally geodesic;*

(1) $(\nabla d\varphi)(\Phi X, \Phi Y) = m'\Phi'(\nabla d\varphi)(X, Y)$, $\forall X, Y \in \Gamma(TM)$,
(2) $(\nabla d\varphi)(\Phi X, \Phi Y) = (\nabla d\varphi)(X, Y)$, $\forall X, Y \in \Gamma(TM)$ *and* $m' \neq 0$.

5. Certain maps between almost poly-Norden manifolds and manifolds endowed with differentiable structures

For maps between differentiable manifolds, authors normally study such maps under certain conditions imposed on the manifolds and maps. A crucial question is that whether there exist such maps under the restrictions. Therefore, in this section, we investigate the existence of holomorphic-like maps from (into) poly-Norden manifolds to manifolds endowed with differentiable structure such as almost Golden structure, almost complex structure, almost product structure and almost contact structure. We show that such maps defined between almost poly-Norden manifolds and manifolds endowed with differentiable structures are constant under some assumptions.

5.1. **Maps between almost poly-Norden manifolds and almost Golden manifolds.** Let $\bar{M}$ be a differentiable manifold. A golden structure on $\bar{M}$ is an $(1,1)$ tensor field P which satisfies the equation

$$P^2 = P + I \tag{5.9}$$

where I is the identity transformation. In this case P is called an almost Golden structure and (M, P) is called almost Golden manifold. We say that the metric g is P compatible if

$$g\,(PX, Y) = g\,(X, PY) \tag{5.10}$$

for all $X, Y \in \Gamma\left(T\bar{M}\right)$. If we substitute PX into X in (2.12) the equation (2.12) may also written as

$$g\,(PX, PY) = g\left(P^2X, Y\right) = g\,((P + I)\,X, Y) = g\,(PX, Y) + g\,(X, Y)$$

The Riemannian metric (2.12) is called $P-$compatible and $\left(\bar{M}, P, g\right)$ is named a Golden Riemannian manifold [5]. It is known[5] that a Golden structure φ is integrable if the Nijenhuis tensor N_φ vanishes. In [10], the authors show that a Golden structure is integrable if and only if $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g.

Theorem 5.1. *Let φ be a smooth map from an almost Golden manifold $\left(\bar{M}, P,\right)$ to an almost poly-Norden manifold (M', Φ, m) such that the condition $d\varphi P = \Phi d\varphi$ is satisfied. Then φ is a constant map if $m \neq \mp\sqrt{5}$.*

Proof. Let $\left(\bar{M}, P\right)$ be an almost Golden manifold and (M', Φ, m) a almost poly-Norden manifold. Then apply Φ to equation $d\varphi P = \Phi d\varphi$ and using (3.1) and (2.11), we get

$$(1-m)\Phi d\varphi(X) = -2d\varphi\,(X) \tag{5.11}$$

for $X \in \Gamma\left(T\bar{M}\right)$. Applying Φ to (5.11) again, we derive

$$(-m^2+m+2)\Phi d\varphi\,(X) = (1-m)d\varphi\,(X) \tag{5.12}$$

Then (5.11) and (5.12) imply that

$$(5-m^2)\Phi d\varphi(X) = 0$$

which gives our assertion.

In a similar way, we have the following result.

Theorem 5.2. *Let φ be a smooth map from an almost poly-Norden manifold (M', Φ, m) to an almost Golden manifold $\left(\bar{M}, P\right)$ such that the condition $d\varphi\Phi = Pd\varphi$ is satisfied. Then φ is a constant map if $m \neq \mp\sqrt{5}$.*

5.2. **Maps between almost poly-Norden manifolds and almost complex manifolds.** Let M' be a $2n-$ dimensional real manifold. An almost complex structure J on M' is a $(1,1)$ tensor field such that

$$J^2 = -I \tag{5.13}$$

where I is the identity transformation. Then (M', J) is called almost complex manifold [18].

Theorem 5.3. *Let φ be a smooth map from an almost poly-Norden manifold $\left(\bar{M}, \Phi, m\right)$ to an almost complex manifold(M', J) such that the condition $d\varphi\Phi = Jd\varphi$ is satisfied. Then φ is a constant map.*

Proof. Let $\left(\bar{M}, \Phi, m\right)$ be an almost poly-Norden manifold and (M', J) an almost complex manifold. Suppose that $\varphi : \bar{M} \to M'$ satisfies $d\varphi\,(\Phi X) = Jd\varphi\,(X)\,, \quad X \in \Gamma\left(T\bar{M}\right)$. Then apply J to above equation and using (2.1) and (2.11), we get

$$d\varphi\,(m\Phi X) - d\varphi\,(X) = -d\varphi\,(X)\,, \quad X \in \Gamma\left(T\bar{M}\right). \tag{5.14}$$

From (5.14) we obtain $mJd\varphi(X) = 0$ which shows that φ is constant due to J is nonsingular.

In a similar way, we have the following result.

Theorem 5.4. *Let φ be a smooth map from an almost complex manifold (M', J) to an almost poly-Norden manifold $\left(\bar{M}, \Phi, m\right)$ such that the condition $d\varphi J = \Phi d\varphi$ is satisfied. Then φ is a constant map.*

5.3. **Maps between almost poly-Norden manifolds and almost product manifolds.** Let N be an $n-$dimensional manifold with a tensor of type (1.1) such that

$$F^2 = I, \tag{5.15}$$

where I is the identity transformation. Then we say that N is an almost product manifold with almost product structure F. We put

$$Q = \frac{1}{2}\left(I + F\right), \quad Q^{'} = \frac{1}{2}\left(I - F\right). \tag{5.16}$$

Then we have

$$Q + Q^{'} = I, \;\; Q^2 = Q, \;\; Q^{'2} = Q^{'}, \;\; QQ^{'} = Q'Q = 0 \tag{5.17}$$

and

$$F = Q - Q'. \tag{5.18}$$

for details, see:[18].

For a holomorphic-like map between poly-norden manifolds and almost product manifolds, we have the following result.

Theorem 5.5. *Let φ be a smooth map from an almost poly-Norden manifold (M', Φ, m) to an almost product manifold $\left(\bar{M}, F\right)$ such that the condition $d\varphi\Phi = F d\varphi$ is satisfied. Then φ is a constant map if $m \neq \mp 2$.*

Proof. Applying F to $d\varphi\Phi = F d\varphi$ and using (3.3) and (5.15) we have

$$md\varphi(\Phi X) = 2d\varphi(X) \tag{5.19}$$

for $X \in \Gamma(TM)$. Applying F to (5.19) and using (3.3) we obtain

$$(m^2 - 2)d\varphi(\Phi X) = md\varphi(X). \tag{5.20}$$

Thus from (5.19) and (5.20) we get

$$(m^2 - 4)d\varphi(X) = 0$$

which gives proof.

In a similar way, we have the following result.

Theorem 5.6. *Let φ be a smooth map from an almost product manifold $\left(\bar{M}, F\right)$ to an almost poly-Norden manifold (M', Φ, m) such that the condition $d\varphi F = \Phi d\varphi$ is satisfied. Then φ is a constant map if $m \neq \mp 2$.*

5.4. **Maps between almost poly-Norden manifolds and almost contact manifolds.** An $n-$dimensional differentiable manifold $\bar{M}$ is said to have an almost contact structure $(\bar{\varphi}, \xi, \eta)$ if it carries a tensor field $\bar{\varphi}$ of type $(1,1)$, a vector field ξ and $1-$ form η on $\bar{M}$ respectively such that

$$\bar{\varphi}^2 = -I + \eta \otimes \xi, \bar{\varphi}\xi = 0, \eta \circ \bar{\varphi} = 0, \eta\left(\xi\right) = 1 \tag{5.21}$$

where I is the identity transformation [2].

Theorem 5.7. *Let φ be a smooth map from an almost contact manifold $\left(\bar{M}, \bar{\varphi}, \eta, \xi\right)$ to an almost poly-Norden manifold (M', Φ, m) such that the condition $d\varphi\bar{\varphi} = \Phi d\varphi$ is satisfied. Then φ is a constant map.*

Proof. Using (3.3) and (5.21) we have

$$m\Phi d\varphi(X) = \eta(X) d\varphi(\xi) \tag{5.22}$$

for $X \in \Gamma(T\bar{M})$. Using the second relation of (5.21) in (5.22) and applying (3.3) we find

$$m\Phi d\varphi(X) = d\varphi(X). \tag{5.23}$$

Using once again (3.3) and the relation $d\varphi\bar{\varphi} = \Phi d\varphi$ we get

$$(m^2 - 1)\Phi d\varphi(X) = m d\varphi(X). \tag{5.24}$$

Thus from (5.23) and (5.24) we obtain $d\varphi(X) = 0$, $\forall X \in \Gamma(T\bar{M})$.

In a similar way, we have the following result.

Theorem 5.8. *Let φ be a smooth map from an almost poly-Norden manifold (M', Φ, m) to an almost contact manifold $\left(\bar{M}, \bar{\varphi}, \eta, \xi\right)$ such that the condition $d\varphi\Phi = \bar{\varphi} d\varphi$ is satisfied. Then φ is a constant map.*

Remark 5.1. *In this paper, we introduce a new manifold defined by a new mean given in* [13]. *As we have seen, this new manifold has rich geometric properties and it is also useful to characterize certain maps. Therefore, we invite readers to explore further geometric properties of this new class.*

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