DECOMPOSITION ON QTAG-MODULES WITH CERTAIN SUBMODULES

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Abstract. A module $M$ over an associative ring $R$ with unity is a QTAG-module if every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules. In this paper, we study some decomposition results on QTAG-modules with certain submodules in terms of the cardinality $g(M)$.

1. Introduction and Terminology

Let $R$ be any ring with unity. A uniserial module $M$ is a module over a ring $R$, whose submodules are totally ordered by inclusion. This means simply that for any two submodules $N_1$ and $N_2$ of $M$, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module $M$ is called a serial module if it is a direct sum of uniserial modules. An element $x \in M$ is uniform, if $xR$ is a non-zero uniform (hence uniserial) module and for any $R$-module $M$ with a unique decomposition series, $d(M)$ denotes its decomposition length.

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A module $M_R$ is called a TAG-module if it satisfies the following two conditions:

(I) Every finitely generated submodule of every homomorphic image of $M$ is a direct sum of uniserial modules.

(II) Given any two uniserial submodules $U$ and $V$ of a homomorphic image of $M$, for any submodule $W$ of $U$, any non-zero homomorphism $f: W \to V$ can be extended to a homomorphism $g: U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

A module $M_R$ satisfying only condition (I) is called a QTAG-module. The study of QTAG-modules was initiated by Singh [11]. This is a very fascinating structure that has been the subject of research of many authors. Different notions and structures of QTAG-modules have been studied, and a theory was developed, introducing several notions, interesting properties, and different characterizations of submodules. Many interesting results have been obtained, but there is still a lot to explore.

Everywhere in the text of the present article; let it be agreed that all the rings are associative with unity ($1 \neq 0$) and modules are unital QTAG-modules. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup \left\{ d\left( \frac{yR}{xR} \right) \mid y \in M, x \in yR \text{ and } y \text{ uniform} \right\}$ are the exponent and height of $x$ in $M$, respectively. $H_k(M)$ denotes the submodule of $M$ generated by the elements of height at least $k$ and $H^k(M)$ is the submodule of $M$ generated by the elements of exponents at most $k$. Let us denote by $M^1$, the submodule of $M$, containing elements of infinite height. As defined in [5], the module $M$ is $h$-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$. The module $M$ is $h$-reduced if it does not contain any $h$-divisible submodule. In other words, it is free from the elements of infinite height. The module $M$ is said to be bounded, if there exists an integer $k$ such that $H_M(x) \leq k$ for every uniform element $x \in M$.

A submodule $N$ of $M$ is $h$-pure [3] in $M$ if $N \cap H_n(M) = H_n(N)$, for every integer $n \geq 0$. A submodule $B \subseteq M$ is a basic submodule [5] of $M$, if $B$ is $h$-pure in $M$, $B = \oplus B_i$, where each $B_i$ is the direct sum of uniserial modules of length $i$ and $M/B$ is $h$-divisible. A submodule $N \subseteq M$ is said to be high [4], if it is a complement of $M^1$ i.e. $M = N \oplus M^1$. The sum of all simple submodules of $M$ is called the socle of $M$ and is denoted by $Soc(M)$. The cardinality of the minimal generating set of $M$ is denoted by $g(M)$. For all ordinals $\alpha$, $f_M(\alpha)$ is the $\alpha^{th}$-Ulm invariant of $M$ (see [6]) and it is equal to $g(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M)))$.
Imitating [8], the submodules $H_k(M), k \geq 0$ form a neighborhood system of zero, thus a topology known as $h$-topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to $h$-topology if $\overline{N} = N$.

It is interesting to note that almost all the results which hold for $TAG$-modules are also valid for $QTAG$-modules [9]. Many results of this paper are the generalization of [10]. Our notations and terminology generally agree with those in [1] and [2].

2. Elementary results

We start here with a recollection of the following notions from [7].

**Definition 2.1.** A basic submodule $B^u$ of a $QTAG$-module $M$ is said to be an upper basic submodule if $g(M/B^u) = \min\{g(M/B) | B \text{ is a basic submodule of } M\}$.

**Definition 2.2.** A basic submodule $B^l$ is said to be a lower basic submodule of $M$ if $g(M/B^l) = \inf\{H_k(M)\}$.

To develop the study, we need to prove some elementary but helpful lemmas.

**Lemma 2.1.** If $M$ is a $QTAG$-module without elements of infinite height such that every basic submodule of $M$ is both an upper and lower basic submodule of $M$, and such that $\inf g(M) = g(M)$, then $M$ cannot be decomposed as $M = M_1 \oplus M_2$, where $M_2$ is a direct sum of uniserial modules, and $g(M_1) < g(M)$.

**Proof.** Suppose such a decomposition of $M$ does exist, and let $B$ be a basic submodule of $M_1$. Now $B \oplus M_2$ is a basic submodule of $M$ and

$$g(M/(B \oplus M_2)) = g(M_1/B) + g(M_2/M_2) = g(M_1/B) \leq g(M_1) < g(M).$$

Since $\inf g(M) = g(M)$ there exists a basic submodule $B'$ of $M$ such that $g(M/B') = g(M)$. But these two facts contradict the hypothesis that every basic submodule of $M$ is both an upper and lower basic submodule of $M$.

**Lemma 2.2.** Let $M$ be a $QTAG$-module without elements of infinite height. Suppose $M = M_1 \oplus M_2$, where $M_2$ is a direct sum of uniserial modules, and suppose that every basic submodule of $M_1$ is both an upper and lower basic submodule of $M_1$. If $\inf g(M_1) = g(M_1)$,
\[ g(M_2) < g(M_1), \text{ and } B \text{ is a basic submodule of } M_1, \text{ then } B \oplus M_2 \text{ is an upper basic submodule of } M. \]

**Proof.** Suppose that \( B \oplus M_2 \) is not an upper basic submodule of \( M \), and let \( B^u \) be an upper basic submodule of \( M \). Let \( M = S \oplus P_1 \) and \( B^u = P_1 \oplus P_2 \) where \( g(S) = \max \{ \aleph_0, g(M/B^u) \} \). If \( g(S) \leq \aleph_0 \) then we get that \( M \) is a direct sum of uniserial modules, and therefore \( M_1 \) is a direct sum of uniserial modules. Thus \( M_1 \) must be bounded since each of its basic submodules is both an upper and lower basic submodule. Therefore \( B = M_1 \) and so \( B \oplus M_2 = M_1 \oplus M_2 \) is an upper basic submodule of \( M \). Now we can write \( M = S \oplus Q_1 \oplus Q_2 \) where \( P_1 = Q_1 \oplus Q_2 \), and \( S \oplus Q_1 \) contains \( M_2 \) and \( g(S \oplus Q_1) < g(M_1) \). But we know that \( M_1 \cong M/M_2 \cong [(S \oplus Q_1)/M_2] \oplus Q_2 \) which contradicts Lemma 2.1 when applied to \( M_1 \), therefore \( B \oplus M_2 \) must be an upper basic submodule of \( M \).

**Lemma 2.3.** Let \( M \) be a QTAG-module without elements of infinite height such that \( M = M_1 \oplus M_2 \), where \( M_2 \) is a direct sum of uniserial modules, \( \text{fin } g(M_1) = g(M_2) \), and every basic submodule of \( M_1 \) is both an upper and lower basic submodule of \( M_1 \). If \( B \) is a basic submodule of \( M_1 \), then \( B \oplus M_2 \) is an upper basic submodule of \( M \).

**Proof.** Suppose that \( B \oplus M_2 \) is not an upper basic submodule of \( M \), and let \( B^u \) be an upper basic submodule of \( M \). As in Lemma 2.2 we can assume that \( g(M/B^u) > \aleph_0 \).

Write \( M = S \oplus P_1 \) and \( B^u = P_1 \oplus P_2 \) where \( g(S) = g(M/B^u) < g(M/(B \oplus M_2)) \leq g(M_1) \).

Now we can write \( M = M_1 \oplus R_1 \oplus R_2 \) where \( M_2 = R_1 \oplus R_2 \) and \( M_1 \oplus R_1 \) contains \( S \) and \( g(R_1 + S) < g(M_2) \). Consider the module \( M_1 \oplus R_1 \). By Lemma 2.2 we know that \( B \oplus R_1 \) is an upper basic submodule of \( M_1 \oplus R_1 \). But \( M_1 \oplus R_1 \) contains \( S \) which is a summand of \( M \) so that we can write \( M_1 \oplus R_1 = S \oplus [(M_1 \oplus R_1) \cap P_1] \). Let \( T = (M_1 \oplus R_1) \cap P_1 \). Now observe that \( g((M_1 \oplus R_1)/(P_2 \oplus T)) = g(S/P_2) \leq g(S) < g(M_1) \). Since \( \text{fin } g(M_1) = g(M_1) \), and every basic submodule of \( M_1 \) is both an upper and lower basic submodule of \( M_1 \) with \( g(M_1/B) = g(M_1) \) which contradicts \( B \oplus R_1 \) being an upper basic submodule of \( M_1 \oplus R_1 \). Thus \( B \oplus M_2 \) must be an upper basic submodule of \( M \).

**Lemma 2.4.** Let \( M \) be an \( h \)-reduced QTAG-module such that every basic submodule of \( M \) is an upper and lower basic submodule of \( M \). Suppose \( M = M_1 \oplus M_2 \), where \( M_2 \) is a direct sum of uniserial modules. Then \( M_1 \) has the property that each of its basic submodules is both an upper and lower basic submodule of \( M_1 \).
Suppose there exists two basic submodules $B_1$ and $B_2$ of $M_1$ such that $g(M_1/B_1) \neq g(M_1/B_2)$. Then $B_1 \oplus M_2$ and $B_2 \oplus M_2$ are basic submodules of $M$ such that $g(M/(B_1 \oplus M_2)) = g(M_1/B_1) \neq g(M_1/B_2) = g(M/(B_2 \oplus M_2))$, and this contradicts the hypothesis on $M$.

**Lemma 2.5.** Let $M$ be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where $\text{fin} \ g(M) = g(M)$, $\text{fin} \ g(M_1) = g(M_1)$, $\text{fin} \ g(M_2) = g(M_2)$, and $M_1$ and $M_2$ have the property that every basic submodule is an upper and lower basic submodule. If $M = M_3 \oplus M_4$ where $g(M_3) < g(M)$, and $M_4$ is a direct sum of uniserial modules, then $g(M) = g(M_1) = g(M_2)$.

**Proof.** Suppose that $g(M_1) < g(M)$, then $g(M_2) = g(M)$. Now we write $M = M_3 \oplus T_1 \oplus T_2$ where $M_3 \oplus T_1$ contains $M_1$, and $g(M_3 \oplus T_1) < g(M)$. This is possible since $g(M_3) < g(M)$, and $g(M_1) < g(M)$. Now notice that $M_2 \cong M/M_1 \cong [(M_3 \oplus T_1)/M_1] \oplus T_2$, and $g((M_3 \oplus T_1)/M_1) \leq g(M_3 \oplus T_1) < g(M) = g(M_2)$. But this contradicts Lemma 2.1 when applied to $M_2$, therefore we have $g(M) = g(M_1) = g(M_2)$.

**Lemma 2.6.** Let $M$ be a QTAG-module without elements of infinite height and suppose that $M = M_1 \oplus M_2$, where $\text{fin} \ g(M_1) = g(M_1)$, and $\text{fin} \ g(M_2) = g(M_2)$, and $g(M) = g(M)$, and every basic submodule of $M_1$ or $M_2$ is both an upper and lower basic submodule. Let $B_1$ and $B_2$ be basic submodules of $M_1$ and $M_2$ respectively. If either $g(M_1) < g(M)$ or $g(M_2) < g(M)$, then $B_1 \oplus B_2$ is an upper basic submodule of $M$.

**Proof.** Assume that $B_1 \oplus B_2$ is not an upper basic submodule of $M$. Let $B^u$ be an upper basic submodule of $M$. Write $M = M_3 \oplus M_4$ where $M_4$ is a direct sum of uniserial modules, and $g(M_3) = \max(\aleph_0, g(M/B^u))$. If $g(M_3) \leq \aleph_0$, then $M$ is a direct sum of uniserial modules. But this means that $M_1$ and $M_2$ are bounded since the bounded direct sum of uniserial modules are the only direct sums of uniserial modules which have the property that every basic submodule is both an upper and lower basic submodule. Thus $M$ is a bounded module and hence only basic submodule which contradicts the assumption that $B_1 \oplus B_2$ is not an upper basic submodule of $M$. Therefore $\aleph_0 < g(M/B^u)$, and $g(M_3) = g(M/B^u)$.

Since $B_1 \oplus B_2$ is not an upper basic submodule of $M$, we get that $g(M/B^u) < g(M)$, but this contradicts Lemma 2.5. Thus $B_1 \oplus B_2$ must be an upper basic submodule of $M$. 

Theorem 3.1. Let $M$ be a QTAG-module without elements of infinite height such that $M = M_1 \oplus M_2$, where $M_2$ is a direct sum of uniserial modules. If $B^u$ is an upper basic submodule of $M_1$, then $B^u \oplus M_2$ is an upper basic submodule of $M$.

Proof. If $M_1$ is a finitely generated, module then $B^u \oplus M_2 = M$ since $B^u$ is basic submodule in $M_1$, thus $B^u \oplus M_2$ is upper basic submodule. Since $B^u$ is an upper basic submodule of $M_1$, we can write, $M_1 = S_1 \oplus T_1$ and $B^u = T_1 \oplus T_2$ where $g(S_1) = \max(\kappa_0, g(M_1/B^u))$, and every basic submodule of $S_1$ is both an upper and lower basic submodule of $S_1$. As in Lemma 2.2, we assume that $g(M_1/B^u) > \kappa_0$. Now if $\text{fin } g(S_1) < g(S_1)$ then we write $S_1 = S_2 \oplus S_3$ and $T_1 = K \oplus S_3$ where $\text{fin } g(S_2) < g(S_2)$. By Lemma 2.4 every basic submodule of $S_2$ is both an upper and lower basic submodule of $S_2$. Thus $M = S_2 \oplus S_3 \oplus T_2 \oplus M_2$ where $S_3 \oplus T_2 \oplus M_2$ is a direct sum of uniserial modules, and hence, by Lemma 2.3 we have that $K \oplus S_3 \oplus T_2 \oplus M_2$ is an upper basic submodule of $M$, but $K \oplus S_3 \oplus T_2 \oplus M_2 = B^u \oplus M_2$.

With the last statement in hand, we establish the following corollaries about decomposition of QTAG-modules.

Corollary 3.1. Let $M$ be a QTAG-module without elements of infinite height such that $M = M_1 \oplus M_2 = M_3 \oplus M_4$, where $\text{fin } g(M_1) = g(M_1)$ and $\text{fin } g(M_3) = g(M_3)$. Suppose that $M_2$ and $M_4$ are direct sums of uniserial modules, and every basic submodule of $M_1$ and $M_3$ is an upper and lower basic submodule. Then $\text{fin } g(M_1) = \text{fin } g(M_3)$.

Proof. Let $B_1$ and $B_2$ be basic submodule of $M_1$ and $M_3$ respectively. Now we have $\text{fin } g(M_1) = g(M_1/B_1) = g(M/(B_1 \oplus M_2)) = g(M/(B_2 \oplus M_4)) = \text{fin } g(M_3)$, since $B_1 \oplus M_2$ and $B_2 \oplus M_4$ are upper basic submodules of $M$ by Theorem 3.1.

Corollary 3.2. Let $M$ be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where $\text{fin } g(M_1) = g(M_1)$ and $g(M_2) < g(M_1)$, and every basic submodule of $M_1$ is both an upper and lower basic submodule of $M_1$. If $B^u$ is an upper basic submodule of $M_2$, and $B$ is a basic submodule of $M_1$, then $B \oplus B^u$ is an upper basic submodule of $M$.

Proof. If $M$ is finitely generated, the proof is trivial. Now we write $M_2 = S \oplus T_1$ and $B^u = T_1 \oplus T_2$ where every basic submodule of $S$ is both an upper and lower basic submodule.
of \( S \). We can also assume that \( \inf g(S) = g(S) \). Consider the module \( M_1 \oplus S \) satisfies the hypothesis of Lemma 2.6, and so \( B \oplus T_2 \) is an upper basic submodule of \( M_1 \oplus S \). Now by Theorem 3.1, we have \( B \oplus T_1 \oplus T_2 = B \oplus B'' \) is an upper basic submodule of \( M \).

And so, we are ready to prove the following.

**Proposition 3.1.** Let \( M \) be a QTAG-module without elements of infinite height. Suppose that \( M = M_1 \oplus M_2 \), where \( \inf g(M_1) = g(M_1) \) and \( \inf g(M_2) = g(M_2) \), \( \inf g(M) = g(M) \), and every basic submodule of \( M_1 \) and \( M_2 \) is both an upper and lower basic submodule. If \( B_1 \) and \( B_2 \) are basic submodules of \( M_1 \) and \( M_2 \) respectively, then \( B_1 \oplus B_2 \) is an upper basic submodule of \( M \).

**Proof.** By Lemma 2.6 we assume that \( g(M_1) = g(M_2) = g(M) \). Suppose that \( B_1 \oplus B_2 \) is not upper basic submodule of \( M \), and let \( B'' \) be an upper basic submodule of \( M \). Now we write \( M = S \oplus Q_1 \) and \( B'' = Q_1 \oplus Q_2 \) where \( S \) has the property that every basic submodule of \( S \) is both an upper and a lower basic submodule of \( S \), and \( g(S) = \max(\aleph_0, g(M/B'')) \). As in the proof of Lemma 2.6 we can assume that \( \aleph_0 < g(S) = g(M/B'') < g(M) \). We may also assume that \( \inf g(S) = g(S) \). Consider the module \( M_1 + S \). Since \( M_1 + S \) contains the modules \( M_1 \) and \( S \), both of which are summands of \( M \), we have \( M_1 + S = M_1 \oplus [(M_1 + S) \cap T] \), and \( M_1 + S = S \oplus [(M_1 + S) \cap Q_1] \). Let \( U = (M_1 + S) \cap T \), and let \( B''_{1u} \) be an upper basic submodule of \( U \). Observing that

\[
g(S) = g(S/Q_2),
\]

\[
= g((S + M_1)/(Q_2 \oplus [(M_1 + S) \cap Q_1])),
\]

\[
= g((M_1 + S)/(B_1 \oplus B''_{1u})),
\]

\[
= g(M_1/B_1) + g(U/B''_{1u})
\]

Now \( g(S) = g(M_1/B_1) + g(U/B''_{1u}) = g(M_1) = g(U/B''_{1u}) \), since \( \inf g(M_1) = g(M_1) \), and every basic submodule of \( M_1 \) is both an upper and a lower basic submodule of \( M_1 \). Thus we have that \( g(M_1) \leq g(S) \), but this a contradiction since \( g(S) < g(M) = g(M_1) \). Therefore \( B_1 \oplus B_2 \) must be an upper basic submodule of \( M \).

**Theorem 3.2.** Let \( M \) be a QTAG-module without elements of infinite height. Suppose that \( M = M_1 \oplus M_2 \), and let \( B''_{1u} \) and \( B''_{2u} \) be upper basic submodules of \( M_1 \) and \( M_2 \) respectively. Then \( B''_{1u} \oplus B''_{2u} \) is an upper basic submodule of \( M \).
Proof. If either $M_1$ or $M_2$ is finitely generated, then by Theorem 3.1, $B_1^u \oplus B_2^u$ is an upper basic submodule in $M$. Now we write $M_1 = S_1 \oplus P_1$ and $B_1^u = P_1 \oplus P_2$ where every basic submodule of $S_1$ is both an upper and a lower basic submodule of $S_1$. Similarly we can write $U = V_1 \oplus T_1$ and $B_2^u = T_1 \oplus T_2$ where every basic submodule of $V_1$ is both an upper and a lower submodule of $V_1$. Now we have $S_1 = S_2 \oplus P_3$ and $P_2 = P_3 \oplus W_1$ where $\text{fin } g(S_2) = g(S_2)$. Similarly we have $V_1 = V_2 \oplus T_3$ and $T_2 = T_3 \oplus W_2$ where $\text{fin } g(V_2) = g(V_2)$. Therefore by Lemma 2.4 we get that $S$ and $B$ are upper basic submodules of $M$. We have two cases to consider.

Case (i). Suppose that $g(K/B_1^u) \leq \aleph_0$, then $M$ is a $\Sigma$-module and $g(M/N) = g(M/K)$. Since $B^u$ is an upper basic submodule of $N$ and $M$ is a $\Sigma$-module, then $B^u = N$, and thus $B^u$ is an upper basic submodule of $M$.

Case (ii). Suppose that $g(K/B_1^u) > \aleph_0$, then $g(M/B^u) > \aleph_0$ and we have $M = S \oplus P_1$ and $B_1^u = P_1 \oplus P_2$ where $g(S) = g(M/B_1^u)$. Now $K$ contains $P_1$ and hence $K = P_1 \oplus T$ where $T = S \cap K$. Let $\phi : M \rightarrow M/M_1$ be the natural quotient map such that $N \cong \phi(N)$, $K \cong \phi(K)$, $B_1^u \cong \phi(B_1^u)$, and $\text{Soc}(\phi(N)) = \text{Soc}(\phi(K))$. Since $K = P_1 \oplus T$ we have $\phi(K) = \phi(B_1^u) \oplus \phi(T)$. Now $\text{Soc}(\phi(B_1^u)) = \bigcup_{i=1}^{\infty} Q_i$ where $Q_i$ is a submodule of elements of bounded height in $\phi(M)$.

For freely use in the sequel, we state the following.

Corollary 3.3. Let $M$ be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where every basic submodule of $M_1$ or $M_2$ is both an upper and a lower basic submodule, and suppose that $\text{fin } g(M) = g(M)$. Then every basic submodule of $M$ is an upper and lower basic submodule of $M$.

Proof. Let $B_1^u$ and $B_2^u$ be upper basic submodules of $M_1$ and $M_2$ respectively. By Theorem 3.2, we have $B_1^u \oplus B_2^u$ is an upper basic submodule of $M$. Notice that $\text{fin } g(M) = \text{fin } g(M_1) + \text{fin } g(M_2) = g(M_1/B_1^u) + g(M_2/B_2^u) = g(M/(B_1^u \oplus B_2^u))$, and we are done.

Now we are able to prove the following.

Theorem 3.3. Let $M$ be an $h$-reduced QTAG-module, and let $N$ be a high submodule of $M$. If $B^u$ is an upper basic submodule of $N$, then $B^u$ is an upper basic submodule of $M$.

Proof. Suppose that $B^u$ is not upper basic submodule in $M$, and let $B_1^u$ be an upper basic submodule of $M$ and $K$ a high submodule of $M$ containing $B_1^u$ with $g(M/B_1^u) < g(M)$. We have two cases to consider.

Case (i). Suppose that $g(K/B_1^u) \leq \aleph_0$, then $M$ is a $\Sigma$-module and $g(M/N) = g(M/K)$. Since $B^u$ is an upper basic submodule of $N$ and $M$ is a $\Sigma$-module, then $B^u = N$, and thus $B^u$ is an upper basic submodule of $M$.

Case (ii). Suppose that $g(K/B_1^u) > \aleph_0$, then $g(M/B^u) > \aleph_0$ and we have $M = S \oplus P_1$ and $B_1^u = P_1 \oplus P_2$ where $g(S) = g(M/B_1^u)$. Now $K$ contains $P_1$ and hence $K = P_1 \oplus T$ where $T = S \cap K$. Let $\phi : M \rightarrow M/M_1$ be the natural quotient map such that $N \cong \phi(N)$, $K \cong \phi(K)$, $B_1^u \cong \phi(B_1^u)$, and $\text{Soc}(\phi(N)) = \text{Soc}(\phi(K))$. Since $K = P_1 \oplus T$ we have $\phi(K) = \phi(B_1^u) \oplus \phi(T)$. Now $\text{Soc}(\phi(B_1^u)) = \bigcup_{i=1}^{\infty} Q_i$ where $Q_i$ is a submodule of elements of bounded height in $\phi(M)$.
and consequently in $\phi(N)$ and $\phi(K)$ since both are $h$-pure in $\phi(M)$. Therefore there exists a basic submodule $B_1$ of $\phi(N)$ such that $B_1 \supset Soc(\phi(B_1^u))$. Let $B_2 = \phi^{-1}(B_1) \cap N$, since $\phi$ is an isomorphism between $N$ and $\phi(N)$ and $B_2$ is a basic submodule of $N$. If $g(N/B_2) \leq \aleph_0$ an argument as in Case (i) would complete the proof. Thus assume that $g(N/B_2) > \aleph_0$ and consider

$$g(N/B_2) = g(\phi(N)/B_1)$$

$$= g(Soc(\phi(N)/B_1))$$

$$= g(Soc(\phi(N))/Soc(B_1))$$

$$= g(Soc(\phi(K))/Soc(B_1))$$

$$= g(Soc(\phi(P_1)) \oplus Soc(\phi(T))/Soc(B_1),$$

but $Soc(B_1)$ contains $Soc(\phi(P_1))$. Hence $g(N/B_2) \leq g(Soc(T)) \leq g(Soc(S)) = g(M/B_1^u)$. Notice that $M/B_2 \cong N/B_2 \oplus M/N$, and hence

$$g(M/B_2) = g(N/B_2) + g(M/N) \leq g(M/B_1^u) + g(M/N) = g(M/B_1^u) + g(M/K),$$

and since $g(M/K) \leq g(M/B_1^u)$, we have $g(M/B_2) \leq g(M/B_1^u) + g(M/B_1^u) = g(M/B_1^u)$. Therefore $B_2$ is an upper basic submodule of $M$. We assume that $B^u$ is not upper basic submodule of $M$, and so $g(M/B^u) > g(M/B_2)$. Notice that $g(M/B^u) = g(M/N) + g(N/B^u)$, and $g(M/B_2) = g(M/N) + g(N/B_2)$, so that $g(N/B^u) > g(N/B_2)$ which contradicts $B^u$ being an upper basic submodule of $N$. Therefore $B^u$ is an upper basic submodule of $M$.

**Corollary 3.4.** Let $M$ be a QTAG-module, and let $N_1$ and $N_2$ be high submodules of $M$, and let $B_1^u$ and $B_2^u$ be upper basic submodules of $N_1$ and $N_2$ respectively. Then $g(N_1/B_1^u) = g(N_2/B_2^u)$.

**Proof.** Follows easily from the proof of the last theorem.

4. **Some extended results**

The purpose of the present section is to extending the results of Theorems 3.1 and 3.2. Several such structural consequences are now presented. In this view we first prove the following.
Theorem 4.1. Let $M$ be an $h$-reduced QTAG-module such that $M = M_1 \oplus M_2$. Let $B_1^u$ and $B_2^u$ be upper basic submodules of $M_1$ and $M_2$ respectively. Then $B_1^u \oplus B_2^u$ is an upper basic submodule of $M$.

Proof. Let $N_1$ and $N_2$ be high submodules of $M_1$ and $M_2$ respectively, which contain $B_1^u$ and $B_2^u$ respectively. Now suppose that $B_1^u$ and $B_2^u$ are upper basic submodules of $N_1$ and $N_2$ respectively. By Theorem 3.2 we know that $B_1^u \oplus B_2^u$ is an upper basic submodule of $N_1 \oplus N_2$, and hence by Theorem 3.3, $B_1^u \oplus B_2^u$ is an upper basic submodule of $M$. Now $g(M/(B_1^u \oplus B_2^u)) = g(M_1/B_1^u) + g(M_2/B_2^u)$, and since $B_1^u$ and $B_2^u$ are upper basic submodules of $N_1$ and $N_2$ respectively, we have by Theorem 3.3, they are basic submodules of $M_1$ and $M_2$ respectively. Thus we know that $g(M_1/B_1^u) = g(M_1/B_1^u)$, and $g(M_2/B_2^u) = g(M_2/B_2^u)$. Therefore $g(M/(B_1^u \oplus B_2^u)) = g(M_1/B_1^u) + g(M_2/B_2^u) = g(M_1/B_1^u) + g(M_2/B_2^u) = g(M/(B_1^u \oplus B_2^u))$, and hence $B_1^u \oplus B_2^u$ is an upper basic submodule of $M$.

Theorem 4.2. Let $M$ be an $h$-reduced QTAG-module such that $M = M_1 \oplus M_2$ where $M_2$ is a direct sum of uniserial modules. Let $B^u$ be an upper basic submodules of $M_1$. Then $B^u \oplus M_2$ is an upper basic submodule of $M$.

Proof. The proof follows easily from Theorem 4.1.

Lemma 4.1. Let $M$ be a QTAG-module such that $M = M_1 \oplus M_2$ be a direct sum of uniserial modules, and suppose that $g(M_2) < g(M)$ and that $\aleph_0 < g(M_2)$ is not a limit cardinal. Let $N$ be an $h$-pure submodule of $M_2$, and let $B_1 \oplus N$ be a basic submodule of $M$ such that $g(M/(B_1 \oplus N)) > g(M_2)$. Then there exists a basic submodule $B_2 \oplus N$ such that $B_2 \oplus N$ contains $B_1 \oplus N$, and $g(M/(B_2 \oplus N)) \leq g(M_2)$.

Proof. Consider $M/B_1 = M_3/B_1 \oplus M_4/B_1$ where $M_4/B_1$ is an $h$-reduced and contain $(B_1 \oplus N)/B_1$, and where $M_3/B_1$ is $h$-divisible. Notice that $M_3 \cap N = 0$ since $M_3 \cap B_1 = B_1$ and $B_1 \cap N = 0$. Thus $M_3 + N = M_3 \oplus N$, and as a submodule of a direct sum of uniserial modules is itself a direct sum of uniserial modules. To show that $M_3 \oplus N$ is a basic submodule, we need only prove that $M_3 \oplus N$ is $h$-pure, but $(M_3 \oplus N)/(B_1 \oplus N) \cong M_3/B_1$ which is $h$-divisible and hence $M_3 \oplus N$ is $h$-pure. Therefore $M_3 \oplus N$ is a basic submodule of $M$ which contains $B_1 \oplus N$, and notice that $g(M/(M_3 \oplus N)) \leq g(M/M_3) = g(M_4/B_1)$, and $g(M_4/B_1) \leq g(M_2)$ since $g(M_2)$ is not a limit ordinal. This completes the proof.

Theorem 4.3. Let $M$ be a QTAG-module without elements of infinite height, and let $B$ be a basic submodule of $M$. Let $B^u$ be an upper basic submodules of $M$, and suppose that
\( g(M/B^u) \) is not a limit cardinal larger than \( \aleph_0 \). Then \( B \) is contained in an upper basic submodule of \( M \).

**Proof.** If \( g(M/B^u) \leq \aleph_0 \), then \( M \) is a direct sum of uniserial modules and hence \( B \) is contained in an upper basic submodule of \( M \), namely, \( M \) itself. Thus we may assume that \( \aleph_0 < g(S) = g(M/B^u) \), \( M = S \oplus P_1 \) where \( B^u = P_1 \oplus P_2 \) and \( g(S) = g(M/B^u) \). Let \( S \) be the homomorphic image of the free module \( T \) with \( h \)-pure kernel \( K \) and we can assume \( g(T) = g(S) \). Now \( (P_1 \oplus T)/K \cong S \oplus P_1 \), and suppose \( (Q_1 \oplus K)/K \cong B \). If \( g(M/B) = g(M/B^u) \) we know that \( B \) is already an upper basic submodule of \( M \) and we are done, so that we can assume that \( g(M/B) > g(M/B^u) = g(S) \). Thus \( g[(T \oplus P_1)/(Q_1 \oplus K)] > g(S) = g(T) \), and by Lemma 4.1 there exists a basic submodule \( Q_2 \oplus K \) containing \( Q_1 \oplus K \) and such that \( g[(T \oplus P_1)/(Q_2 \oplus K)] = g(T) \). Let \( R \cong (Q_2 \oplus K)/K \). We know that \( R \) is a basic submodule of \( M \) which contains \( B \) and \( R \) is upper basic submodule of \( M \) since \( g(M/R) = g[(T \oplus P_1)/(Q_2 \oplus K)] = g(S) = g(M/B^u) \).

**Theorem 4.4.** Let \( M \) be an \( h \)-reduced \( QTAG \)-module and let \( B \) be a basic submodule of \( M \). If there exists a high submodule \( N \) of \( M \) which contains \( B \), and an upper basic submodule \( B^u \) of \( N \) containing \( B \), then \( B \) is contained in an upper basic submodule of \( M \).

**Proof.** If \( B^u \) is an upper basic submodule of \( N \), then \( B^u \) is an upper basic submodule of \( M \) by Theorem 3.3.

Let \( \mathfrak{B} \) be the class of \( QTAG \)-modules which have the property that every basic submodule is contained in an upper basic submodule.

**Theorem 4.5.** The class \( \mathfrak{B} \) contains all \( QTAG \)-modules which are direct sum of an \( h \)-divisible and a bounded module.

**Proof.** This follows immediately from the fact that such modules have only basic submodule, and it is by necessity an upper basic submodule.

Recall that a module \( M \) is a \( \Sigma \)-module (see [4]) if some its high submodule is a direct sum of uniserial modules.

**Theorem 4.6.** The class \( \mathfrak{B} \) contains all \( \Sigma \)-modules.

**Proof.** Let \( M \) be \( \Sigma \)-module, and \( B \) be a basic submodule of \( M \). Now \( B \) can be embedded in a high submodule of \( N \) of \( M \), and since \( M \) is a \( \Sigma \)-module we get that \( N \) is an upper basic submodule of \( M \).
\textbf{Theorem 4.7.} Let $M$ be a $\text{QTAG}$-module without elements of infinite height. Suppose that $\text{fin } g(M)$ is equal to its cardinality, and that $B$ is a basic submodule of $M$. If $g(M/B)$ is equal to the cardinality of $\overline{M}$, the closure of $M$, then $M \in \mathcal{B}$.

\textbf{Proof.} This follows from Theorem 4.3.

\textbf{Corollary 4.1.} The class $\mathcal{B}$ contains all closed modules.

5. Open problems

In closing, we pose the following questions of interest:

\textbf{Problem 5.1.} If $M$ is an $h$-reduced $\text{QTAG}$-module such that $M = \sum_{\alpha \in I} M_\alpha$, and $B_\alpha$ is an upper basic submodule of $M_\alpha$, then is it true that $\sum_{\alpha \in I} B_\alpha$ is upper basic submodule of $M$?

\textbf{Problem 5.2.} Does the class $\mathcal{B}$ defined above indeed contains all $h$-reduced $\text{QTAG}$-modules?

\textbf{References}